Sven de Vries

Discrete Tomography, Packing and Covering, and Stable Set Problems

Polytopes and Algorithms

Herbert Utz Verlag · Wissenschaft
München

Die Deutsche Bibliothek – CIP-Einheitsaufnahme

Ein Titeldatensatz für diese Publikation ist
bei Der Deutschen Bibliothek erhältlich

Zugleich: Dissertation, München, Techn. Univ., 1999

ISBN 3-89675-688-5

Printed in Germany

Herbert Utz Verlag GmbH, München
Tel.: 089/277791-00 – Fax: 089/277791-01

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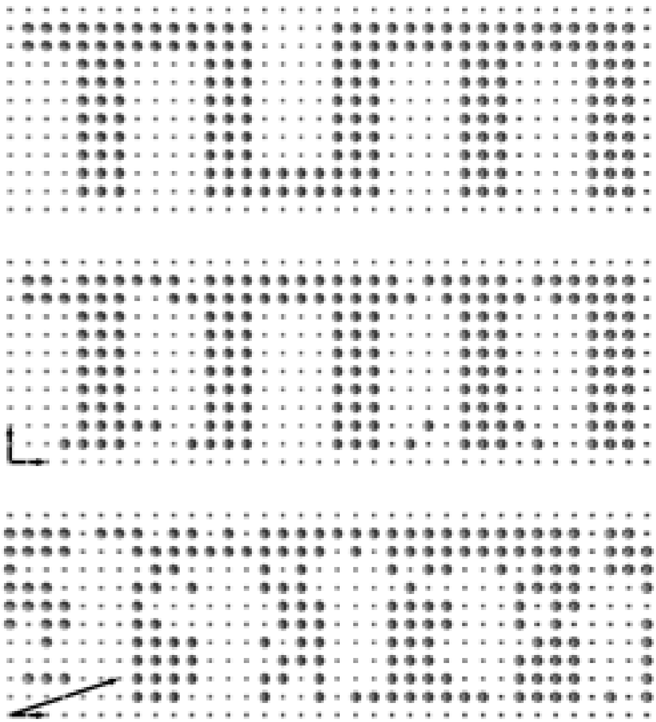


To Daniela and my parents

Contents

Deutschsprachige Übersicht über die Dissertation

In der vorliegenden Dissertation werden Probleme studiert, die durch die Anforderung der Materialwissenschaften, die Oberfläche von Halbleiter-Chips genauer zu analysieren, motiviert sind. Eine neue, kürzlich von Physikern entwickelte Methode erlaubt es, aus elektronenmikroskopischen Aufnahmen von winzigen Kristallen die Anzahl der Atome auf den Atomsäulen in der jeweils verwendeten Projektionsrichtung zu berechnen. Das wichtige Problem, die genaue räumliche Position der Atome auf den Säulen zu bestimmen, löst ihre Methode aber nicht. In dieser Dissertation werden Algorithmen vorgestellt, die eine 3-dimensionale Konfiguration rekonstruieren, die die gegebenen Meßdaten erfüllt. Dann wird von erfolgreichen Computer-unterstützten Experimenten berichtet, Konfigurationen exakt oder approximativ zu rekonstruieren. Diese Rekonstruktionsalgorithmen könnten sich als sehr wichtig erweisen, um der Halbleiter-Industrie zu erlauben, die Produktionsbedingungen von Halbleiter-Chips genauer abzustimmen. Schließlich werden Fragestellungen untersucht, die dem Rekonstruktionsproblem ähnlich sind.

In dieser Zusammenfassung wird zunächst eine Überblick über das Gebiet der Diskreten Tomographie gegeben und die Diskrete Tomographie in das weitere Feld der Tomographie eingeordnet. Nach der Beschreibung mathematisch verwandter Probleme folgt ein ausführlicher Überblick über die in dieser Dissertation erzielten Resultate.

(Diskrete) Tomographie

Das grundlegende Problem der Tomographie ist es, eine unbekannte Funktion f, die eine Menge auf die nichtnegativen reellen Zahlen abbildet, zu rekonstruieren. Über f ist lediglich

1. die Projektion von f auf verschiedene Unterräume,
2. die Summe von f über verschiedene Unterräume oder
3. das Integral von f über verschiedene Unterräume

bekannt.

Ihre klassische Anwendung findet Tomographie in dem Problem, eine 3-dimensionale Rekonstruktion der unbekannten Gewebedichte im menschlichen Körper aus den Röntgenaufnahmen aus einigen hundert Richtungen (im Fall der *Computer Tomographie* (CT)) zu bestimmen. Mathematisch bedeutet dies, bei planaren Problemen eine Radon-Transformation (siehe Radon [RAD17]) und bei höherdimensionalen Problemen eine X-ray-Transformation zu invertieren. Daher werden für diese Probleme insbesondere Methoden der Numerischen Analysis, der Funktionalanalysis sowie der Fourier-Analysis angewendet. Das Rekonstruktionsproblem wird in dieser Anwendung beherrscht, wie jeder weiß, der sich schon einmal einer Computer Tomographie unterzogen hat. Trotzdem ist es natürlich immer wünschenswert, noch bessere Methoden zu finden, die mehr Details aus weniger Daten in kürzerer Zeit bestimmen können. Dieses ist jedoch hier nicht das Ziel.

Andere Klassen von Tomographie-Problemen ergeben sich, wenn die Klasse der zu rekonstruierenden Funktionen, ihre Urbilder oder ihre Bildbereiche eingeschränkt werden. Im folgenden werden die zwei wichtigsten Klassen vorgestellt.

Die erste Klasse ergibt sich, indem die Menge der zu rekonstruierenden Funktionen eingeschränkt wird auf Funktionen, die zugleich charakteristische Funktionen eines geometrischen Objektes im $\mathbb{R}^2$ oder $\mathbb{R}^3$ sind. Dieses ist das Gebiet der *Geometrischen Tomographie;* als Quelle vieler interessanter und aktueller Resultate hierzu eignet sich das Buch von Gardner [GAR95]. Häufig wird die Menge der zulässigen Funktionen weiter eingeschränkt auf charakteristische Funktionen *konvexer Mengen.* Ein bemerkenswertes Problem in diesem Zusammenhang ist die Frage nach der Anzahl der nötigen Richtungen, die in der Ebene einen gegebenen konvexen Körper eindeutig bestimmen. Dieses Problem wurde von Giering [GIE63] gelöst, der zeigte, daß es für jeden solchen Körper drei Richtungen gibt, so daß er durch X-rays in diese drei Richtungen eindeutig bestimmt ist.

Die zweite wichtige Klasse von Tomographie-Problemen sind solche der *Diskreten Tomographie*, die sich ergeben, wenn der Definitionsbereich und das Bild der unbekannten Funktion diskrete Mengen sind. Dabei geht es darum, eine endliche Teilmenge eines Gitters zu rekonstruieren, wozu lediglich die Anzahl ihrer Punkte auf Linien parallel zu wenigen Gittergeraden bekannt ist; jede dieser Zahlen entspricht natürlich der Summe der charakteristischen Funktion der Menge entlang einer Geraden.

Der Begriff der "Discrete Tomography" wurde von Larry Shepp geprägt, als er 1994 ein DIMACS Mini-Symposium mit diesem Titel organisierte. Trotzdem ist die Diskrete Tomographie älter, als das Datum der Namensgebung vermuten läßt. Die ersten Ergebnisse wurden lediglich in anderer Terminologie formuliert, zum Beispiel als Fragen über meßbare Mengen (siehe Lorentz [Lor49]), oder als Fragen über binäre Matrizen (siehe Ryser [Rys57] und [Rys63, Kapitel 6]), oder als Fragen über die endliche Radon-Transformation (siehe Bolker [Bol87]; dort wird die Invertierbarkeit der affinen, der projektiven und der k-Mengen Transformation untersucht).

Später wurde die Frage der eindeutigen Bestimmbarkeit für konvexe Mengen (das heißt hier genauer: *konvexe Gittermengen*) in der diskreten Tomographie gestellt. Sie wurde von Gardner und Gritzmann [GG97] dahingehend beantwortet, daß 4 geeignete Gitterrichtungen ausreichen, jede beliebige konvexe Gittermenge eindeutig festzulegen. Gleichermaßen reichen auch 7 beliebige, paarweise nicht parallele Gitterrichtungen aus.

Nach vielen Jahren, in denen strukturelle Fragen untersucht wurden und Algorithmen für zwei Richtungen (z.B. [Rys57, Lor49]) oder für sehr viele Richtungen (siehe [Bol87]) betrachtet wurden, erhielt das Gebiet einen zusätzlichen Impuls durch eine neue und wichtige Anwendung in den Materialwissenschaften.

Diese neue Hauptanwendung ergibt sich aus einer neuen Analysemethode namens QUANTITEM (von Schwander, Kisielowski, Baumann, Kim und Ourmazd [SKB+93] und von Kisielowski, Schwander, Baumann, Seibt, Kim und Ourmazd [KSB+95]) für Bilder der hochauflösenden Transmissionselektronenmikroskopie. QUANTITEM erlaubt, unter geeigneten Bedingungen, die Anzahl der Atome in den Atomsäulen eines winzigen Kristalls zu *zählen.* Diese Methode eignet sich besonders dazu, durch die amorphe Silizium-Oxid-Schicht hindurch direkt Eigenschaften einer Silizium-Scheibe zu bestimmen. Die Fähigkeit, die verborgene Oberfläche der Silizium-Scheibe zu studieren, ist besonders wichtig für die Halbleiter-Industrie, um die Produktionsbedingungen noch feiner abstimmen zu können. Aber durch die zugrunde liegende Meßtechnik werden neue Anforderungen an die Rekonstruktionsmethoden gestellt:

1. es können nur Messungen in bis zu 5 verschiedene Richtungen gemacht werden, denn der Schaden an der Probe durch den hochenergetischen Elektronenstrahl kann nur bis zu 5 Messungen vernachlässigt werden;

2. das Mikroskop erlaubt nur einen beschränkten Schwenk-Winkel;
3. die Analyse durch QUANTITEM und die Auflösung des Kamera-Detektors (CCD) erlaubt nur Messungen entlang sogenannter Zonen-Axen mit niedrigem Index;
4. die Probe ist möglicherweise nicht konvex.

Infolge dieser neuen Entwicklung werden mittlerweile in der Forschungsliteratur der Diskreten Tomographie algorithmische Methoden verstärkt untersucht. Einerseits gibt es Studien, die sich mit der $\mathbb{NP}$-Schwere verschiedener Rekonstruktionsprobleme befassen (siehe Gardner, Gritzmann und Prangenberg [GGP99], Woeginger [Woe96], Gritzmann, Prangenberg, de Vries und Wiegelmann [GPVW98], Barcucci, Del Lungo, Nivat und Pinzani [BLNP96]). Dann gibt es Aufsätze, in denen polynomial lösbare Aufgaben, wie das Rekonstruktionsproblem für zwei Richtungen, untersucht werden, siehe Anstee [Ans83]. Schließlich sind die experimentellen Arbeiten wie beispielsweise von Salzberg, Rivera-Vega und Rodríguez [SRVR98], und Matej, Herman und Vardi [MHV98] zu erwähnen.

Viele Ergebnisse dieser Dissertation liegen in diesem algorithmisch motivierten Bereich der diskreten Tomographie.

Packungen, Überdeckungen und Stabile Mengen

Eine sehr natürliche Formulierung des Rekonstruktionsproblems der Diskreten Tomographie ergibt sich durch die Interpretation als Fragemengen. Die Punkte auf jeder Geraden einer gegebenen Instanz ergeben eine Fragemenge, und die Messung entlang dieser Geraden assoziiert eine natürliche Zahl mit ihr. Das Problem ist nun, eine Teilmenge der Grundmenge zu finden, die jede der Fragemengen in der entsprechenden Anzahl von Elementen schneidet. Dieses Frageproblem hat die Form eines verallgemeinerten Mengen-Partitionierungsproblems. Da die Rekonstruktion von Konfigurationen von mindestens 3 Messungen $\mathbb{NP}$-schwer ist (siehe [GGP99]), ist es plausibel, das Problem zu relaxieren und die mit den Fragemengen assoziierten Zahlen nur als obere Schranken (oder untere Schranken) für die Größe des Durchschnitts der Fragemenge und der zu bestimmenden Menge zu interpretieren. Dann besteht die Aufgabe darin, eine bezüglich der Bedingungen möglichst große (bzw. möglichst kleine) Menge zu finden. Diese beiden Relaxationen haben die Form eines verallgemeinerten Mengen-Packungs- bzw. Mengen-Überdeckungsproblems.

Die Klasse der Mengen-Partitionierungsprobleme wird häufig zur Modellierung des Problems, Flugzeugbesatzungen in Arbeitsschichten einzuteilen, verwendet (siehe Hoffman und Padberg [HP93]). Eine weitere Anwendung finden diese Probleme in der Disposition verschiedener Verkehrsmittel (siehe Tesch [Tes94] und Borndörfer [Bor97]).

Die Terminologie der Partitionierungsprobleme erlaubt es, diese Anwendungen sehr einfach zu modellieren. Aber in den Lösungsalgorithmen wird normalerweise ausgenutzt, daß eine Lösung des Partitionierungsproblems auch als simultane Lösung eines Packungs- und eines Überdeckungsproblems aufgefaßt werden kann. Diese Vorgehensweise erklärt, warum es wichtig ist, zusätzlich Packungs- und Überdeckungsprobleme zu untersuchen. Im Kapitel 4 werden neue Methoden, um effizient gute approximative Lösungen für verallgemeinerte Mengen-Packungs- und Überdeckungsprobleme zu berechnen, vorgestellt.

Stabile-Mengen-Probleme sind ein (vermeintlicher) Spezialfall von Mengen-Packungsproblemen. Bei ihnen geht es darum, eine möglichst große Teilmenge von nicht benachbarten Knoten eines Graphen zu finden.

Es mag überraschend erscheinen, daß jedes Mengen-Packungsproblem sich in ein Stabile-Mengen-Problem überführen läßt. Die zugrundeliegende Transformation basiert auf der Idee, die Fragemengen als Cliquen des Graphen aufzufassen.

Überblick und Hauptergebnisse

In diesem Abschnitt wird ein Überblick über die verschiedenen Kapitel gegeben. Der Hauptteil mit den wichtigsten Ergebnissen besteht aus den Kapiteln 3–7. Die Kapitel sind im wesentlichen eigenständig; lediglich die gemeinsamen Definitionen und Grundlagen sind in Kapitel 2 zusammenhängend dargestellt. Ein Gesamtüberblick über den Zusammenhang der einzelnen Gebiete ist auf Seite xx in der Tabelle 0.1 dargestellt.

Kapitel 2: Zunächst wird eine gründliche Einführung in die Physik der Meßmethode, die der Diskreten Tomographie zugrunde liegt, gegeben und es wird erklärt, wie Bilder in hochauflösenden Transmissionselektronenmikroskopen entstehen. Dann wird erläutert, wie QUANTITEM Atome zählen kann. Schließlich wird über neue Ergebnisse berichtet, die sich ergeben, indem zunächst die mikroskopische Aufnahme der winzigen Probe simuliert wurde, und das sich ergebende Bild dann mit einer Implementation von QUANTITEM ausgewertet wurde, um die Atomanzahlen auf den Säulen zurückzugewinnen. Hierbei zeigt sich, daß die Zahlen (bis auf Rauschen) in gutem Einklang mit den Vorhersagen der Theorie stehen. Damit wird ein weiterer Beleg angegeben, daß es tatsächlich möglich ist, die Atomanzahlen aus solchen Bildern zu rekonstruieren.

Obgleich das Mikroskop in zwei Richtungen geschwenkt werden kann, steht für die Aufnahmen nur eine Richtung zur Verfügung, denn mit der anderen muß auf das Objekt gezielt werden. Somit liegen alle Meßrichtungen in einer einzigen Ebene, und das ursprünglich 3-dimensionale Rekonstruktionsproblem zerfällt in eine Serie von unabhängig lösbaren, 2-dimensionalen Rekonstruktionsproblemen. Schließlich wird das Rekonstruktionsproblem formal beschrieben.

Kapitel 3: In diesem Abschnitt wird der polyedrische Ansatz benutzt, um die algorithmischen Probleme der Diskreten Tomographie zu modellieren. Diese Probleme sind: *Rekonstruktion*, *Eindeutigkeit* und *Invarianz.* Das *Tomographie-Polytop* wird definiert, und einige seiner gültigen Ungleichungen und Facetten werden studiert.

Algorithmen werden untersucht, um das Rekonstruktionsproblem für 3 Richtungen mit Methoden der ganzzahligen Optimierung zu lösen, und dann werden Ergebnisse einer konkreten Implementation vorgestellt. Probleme der Größe 70×70 lassen sich im Durchschnitt in 9 Minuten lösen; selbst Probleme der Größe 100×100 sind lösbar. In Anbetracht dieser erfolgversprechenden Ergebnisse wird dann ein Algorithmus vorgeschlagen, der in jedem Schritt entweder die Lösung oder eine gute untere Schranke findet. Diese Schranke kann dann benutzt werden, um eine verletzte Ungleichung zu finden. Die Berechnung der verletzten Ungleichung erfordert die Lösung eines sehr viel kleineren, allerdings $\mathbb{NP}$-schweren, Unterproblems.

Kapitel 4: Obwohl die Methoden des vorhergehenden Kapitels geeignet sind, Probleme der Größe 100×100 zu lösen, ist absehbar, daß für die Praxis auch deutlich größere Probleme gelöst werden müssen. Da es aber zur Zeit unmöglich ist, Probleme der Größenordnung 500×500 exakt zu lösen, werden in diesem Kapitel approximative Lösungsverfahren von Relaxationen studiert. Es zeigt sich, daß diese Relaxationen die Form von verallgemeinerten Mengen-Packungs- und Überdeckungsproblemen haben. Für die beiden verallgemeinerten Problemklassen werden sowohl einfache Greedy-Algorithmen als auch kompliziertere iterative Verbesserungs-Algorithmen angegeben. Für ihre Güte ergeben sich Schranken, die im Fall der Packungsprobleme scharf sind.

In der physikalischen Anwendung sind die Tomographie-Probleme normalerweise dicht, d.h., circa die Hälfte der Kandidatenpositionen ist in einer Lösung besetzt. Daher werden das Packungs- und das Überdeckungsproblem für dichte Instanzen studiert. Für diese Probleme wird ein *polynomiales Approximationsschema* konstruiert. Dieses Resultat ist eher von theoretischem Interesse, da die involvierten Konstanten nicht klein sind.

Schließlich werden die Approximations-Algorithmen für Diskrete Tomographie spezialisiert. Die sich ergebenden Approximationsgüten sind für Tomographie-Probleme neu. Es werden Beispiele dafür angegeben, daß die Schranken scharf sind. Eine Implementation der auf der Packungsrelaxation basierenden Algorithmen ist sehr erfolgreich. Sie ist nochmals deutlich besser als ihre bereits guten theoretischen Schranken. In der Anwendung zeigt sich, daß beispielsweise der beste Algorithmus mindestens 99% der Atome in die Konfiguration packt.

Kapitel 5: Nachdem im vorhergehenden Kapitel Verallgemeinerungen der Diskreten Tomographie studiert wurden, wird in diesem Kapitel, als eine weitere Anwendung der Packungsprobleme, das Stabile-Mengen-Problem untersucht. Früher wurde es als eigenständiges Objekt studiert, aber heute ist es vor allem wichtig, weil es eine der beiden Relaxationen von Mengen-Partitionierungsproblemen ist. Das Stabile-Mengen-Problem und das Mengen-Packungsproblem sind, wie wir wissen, gleichwertig.

Der polyedrische Ansatz für Mengen-Partitionierungsprobleme wird häufig bevorzugt, da dadurch die gleichzeitige Ausnutzung des Wissens über beide Relaxationen möglich wird. Die polyedrische Beschreibung des Stabile-Mengen-Problems wird studiert. Klassen von Facetten oder gültigen Ungleichungen erhalten erst dann praktischen Wert, wenn das zugehörige Separationsproblem gelöst werden kann. Hierbei stellt das Separationsproblem die Aufgabe dar, zu einer gegebenen fraktionellen Lösung eine verletzte Ungleichung zu finden. Daher wird im folgenden dem Separationsproblem besondere Aufmerksamkeit geschenkt. Für die Klasse der Anti-Netz-Ungleichungen, die von L. E. Trotter Jr. [Tro75] eingeführt wurden, werden die folgenden Fragen beantwortet:

1. Was läßt sich über das Separationsproblem der Anti-Netz-Ungleichungen beweisen?
2. Wie *stark* sind die Anti-Netz-Ungleichungen?
3. Welche gemeinsame Verallgemeinerung erlauben die Anti-Netz- und die verallgemeinerten Rad-Ungleichungen?

Es ist sehr überraschend, daß über das für die Anwendungen wichtige Separationsproblem der Anti-Netz-Ungleichungen seit ihrer Entdeckung 1975 nichts herausgefunden wurde (nach meinem Kenntnisstand). Dieses ist um so bemerkenswerter, als daß sie sowohl die ungeraden Kreis-Ungleichungen als auch die Cliquen-Ungleichungen als verschiedene Extremfälle enthalten. Für die ungeraden Kreis-Ungleichungen ist eine Separations-Methode von Grötschel, Lovász und Schrijver [GLS93] bekannt; andererseits ist das Separationsproblem für die Cliquen-Ungleichungen $\mathbb{NP}$-schwer, obgleich die Cliquen-Ungleichungen in der polynomial separierbaren Klasse der *orthonormalen Repräsentations-Ungleichungen* [GLS93, 9.3.2] enthalten sind. Mir ist kein Fall bekannt, in dem die Anti-Netz-Ungleichungen in einem Lösungsprogramm für ganzzahlige Programme benutzt wurden. Ein wichtiger Vorteil der Anti-Netz-Ungleichungen ist, daß sie für jedes $k \geq 2$ Ungleichungen enthalten, deren Support-Graph k-zusammenhängend ist. Dadurch sind sie hilfreich in Graphen, die stärker zusammenhängend sind. Die in Anwendungen häufig genutzten ungeraden Kreis-Ungleichungen hingegen sind nur 2-zusammenhängend, und Zweizusammenhang des Support-Graphen ist bereits eine notwendige Bedingung für jede Facette; hier haben Anti-Netz-Ungleichungen ihre besondere Stärke. Es wird eine Hierarchie von Ungleichungsklassen angegeben, die alle Anti-Netz-Ungleichungen umfaßt. Eine Separations-Methode wird präsentiert, die für die Ungleichungen jeder Hierarchie-Ebene in polynomialer Zeit arbeitet. Dieses Ergebnis ist bestmöglich, da andererseits bewiesen wird, daß das Separationsproblem der Anti-Netz-Ungleichungen im allgemeinen $\mathbb{NP}$-schwer ist.

Es werden weiterhin untere Schranken für die *Lovász-Theta-Funktion* [LOV79] für Anti-Netze angegeben, und damit wird bewiesen, daß die Anti-Netz-Ungleichungen nicht von der Klasse der orthonormalen Repräsentations-Ungleichungen impliziert werden.

Eine neue Klasse von gültigen Ungleichungen wird konstruiert, die die Klassen der Anti-Netz-Ungleichungen und der verallgemeinerten Rad-Ungleichungen (siehe Cheng und Cunningham [CC97]) gemeinsam verallgemeinert. Für ihr Separationsproblem wird ein *Algorithmus* vorgestellt. Schließlich werden die Facetten unter den echten Anti-Netz-Rad-Ungleichungen vollständig charakterisiert, indem Graphen-Kompositionen verwendet werden, die Facetten-erhaltend sind.

Kapitel 6: In diesem Kapitel wird ein anderer Aspekt des Polytops der stabilen Mengen studiert. Es ist mitunter sehr mühsam, für eine neue Klasse gültiger Ungleichungen zu beweisen, daß sie Facetten induziert. Dieser Mühsal wird in der Literatur häufig begegnet, indem Operationen benutzt werden, die den Facetten-Beweis für eine komplizierte Konfiguration auf denjenigen für eine einfachere Konfiguration reduzieren. Beispiele hierfür sind:

1. Chvátals [Chv75] Substitution eines Graphen in einen Knoten eines anderen Graphen und
2. Cunninghams [Cun82] Kompositions-Methode.

Einige neue Resultate von Borndörfer und Weismantel [BW97] (siehe auch [Bor97, Chap. 2]) haben mich motiviert, die neue Graphen-Operation der *Partiellen Substitution* zu definieren, die Chvátals und Cunninghams Operationen verallgemeinert. Bedingungen werden angegeben, die garantieren, daß nach der partiellen Substitution von zwei Facetten sich wieder eine Facette ergibt. In Verallgemeinerung einer Idee aus [BW97] wird für eine Klasse von Ungleichungen, die leicht mit der neuen Methode konstruierbar sind, ein polynomialer Separations-Algorithmus konstruiert.

Kapitel 7: Abschließend werden zwei verschiedene Modelle verglichen, die zur Lösung des Rekonstruktionsproblems der Diskreten Tomographie in der Literatur vorgeschlagen wurden. Im ersten Modell ist die einzige Anforderung, daß die Meßdaten erfüllt werden. Im zweiten Modell wird darüberhinaus verlangt, daß alle Linien einer Richtung das Objekt in einem Intervall schneiden. Für beide Modelle werden Algorithmen verwendet, die eine exakte Lösung liefern, welche sich möglichst stark von der Ausgangskonfiguration unterscheidet. Mit diesen Algorithmen läßt sich vergleichen, wieviel Information die verschiedenen Modelle enthalten und zur Verfügung stellen. Beide Algorithmen wurden dann benutzt, um zufällig generierte Instanzen zu lösen. Obwohl klar ist, daß das zweite Modell—*zeilenkonvex* genannt—weniger Mehrdeutigkeit erlaubt als das weniger restriktive erste Modell, so ist doch überraschend, wie stark die Mehrdeutigkeit im ersten Modell ist. Gleichermaßen bemerkenswert ist, daß im zeilenkonvexen Modell, insbesondere im Fall der für die Anwendung besonders wichtigen hohen Dichte, nahezu keine Mehrdeutigkeiten auftreten. Es ist allerdings nicht völlig klar, ob in der Praxis wirklich Zeilenkonvexität der Probe angenommen werden darf.

Zur besseren Übersicht für den Leser gebe ich schließlich eine tabellarische Übersicht über den Zusammenhang der einzelnen Problemfelder (in den Zeilen) und des Lösungsansatzes (in den Spalten) zusammen mit der Angabe des relevanten Abschnittes. Siehe hierzu Tabelle 0.1.

	Polytope	Algorithmen
Diskrete Tomographie	3	2.5, 3.6, 4.6, 7
Packen und Überdecken	3.2	4
Stabile-Mengen-Probleme	5, 6	4.5, 5.3–5.7, 6.3

TABELLE 0.1. Beziehung der verschiedenen Abschnitte dieser Dissertation und der zugehörigen Gebiete.

Bibliographie

[Ans83] R. P. Anstee, *The network flows approach for matrices with given row and column sums*, Discrete Math. **44** (1983), 125–138.

[BLNP96] E. Barcucci, A. Del Lungo, M. Nivat, and R. Pinzani, *Reconstructing convex polyominoes from their horizontal and vertical projections*, Theor. Comput. Sci. **155** (1996), 321–347.

[Bol87] E. D. Bolker, *The finite radon transform*, Contemp. Math. **63** (1987), 27–50.

[Bor97] R. Borndörfer, *Aspects of set packing, partitioning, and covering*, Ph.D. thesis, Technische Universität Berlin (1997).

[BW97] R. Borndörfer and R. Weismantel, *Set packing relaxations of some integer programs*, Technical Report 97-30, Konrad-Zuse-Zentrum (1997).

[CC97] E. Cheng and W. H. Cunningham, *Wheel inequalities for stable set polytopes*, Math. Program. **77** (1997), 389–421.

[Chv75] V. Chvátal, *On certain polytopes associated with graphs*, J. Comb. Theory, Ser. B **18** (1975), 138–154.

[Cun82] W. H. Cunningham, *Polyhedra for composed independence systems*, Ann. Discrete Math. **16** (1982), 57–67.

[Gar95] R. Gardner, *Geometric Tomography*, volume 58 of *Encyclopedia of Mathematics and its Applications*, Cambridge University Press, Cambridge (1995).

[GG97] R. J. Gardner and P. Gritzmann, *Discrete tomography: determination of finite sets by X-rays*, Trans. Am. Math. Soc. **349** (1997), 2271–2295.

[GGP99] R. J. Gardner, P. Gritzmann, and D. Prangenberg, *On the computational complexity of reconstructing lattice sets from their X-rays*, Discrete Math. **202** (1999), 45–71.

[Gie63] O. Giering, *Bestimmung von Eibereichen und Eikörpern durch Steiner-Symmetrisierungen*, Bayer. Akad. Wiss., Math.-naturw. Kl., Sitzungsber. (63), 225–253.

[GLS93] M. Grötschel, L. Lovász, and A. Schrijver, *Geometric algorithms and combinatorial optimization*, volume 2 of *Algorithms and Combinatorics*, Springer, Berlin/Heidelberg, 2. edition (1993).

[GPVW98] P. Gritzmann, D. Prangenberg, S. de Vries, and M. Wiegelmann, *Success and failure of certain reconstruction and uniqueness algorithms in discrete tomography*, Int. J. Imaging Syst. Technol. **9** (1998), 101–109.

[HP93] K. L. Hoffman and M. Padberg, *Solving airline crew scheduling problems by branch-and-cut*, Manage. Sci. **39** (1993), 657–682.

[KSB$^+$95] C. Kisielowski, P. Schwander, F. H. Baumann, M. Seibt, Y. Kim, and A. Ourmazd, *An approach to quantitative high-resolution transmission electron microscopy of crystalline materials*, Ultramicroscopy **58** (1995), 131–155.

[Lor49] G. G. Lorentz, *A problem of plane measure*, Amer. J. Math. **71** (1949), 417–426.

[Lov79] L. Lovász, *On the Shannon capacity of a graph*, IEEE Trans. Inf. Theory **25** (1979), 1–7.

[MHV98] S. Matej, G. T. Herman, and A. Vardi, *Binary tomography on the hexagonal grid using gibbs priors*, Int. J. Imaging Syst. Technol. **9** (1998), 126–131.

[Rad17] J. Radon, *Über die Bestimmung von Funktionen durch ihre Integralwerte längs gewisser Mannigfaltigkeiten*, Berichte Sächsische Akademie der Wissenschaften, Math.-Phys. Kl. **69** (1917), 262–267.

[Rys57] H. J. Ryser, *Combinatorial properties of matrices of zeros and ones*, Canad. J. Math. **9** (1957), 371–377.

[Rys63] —, *Combinatorial Mathematics*, Mathematical Association of America and Quinn & Boden, Rahway, NJ (1963).

[SKB+93] P. Schwander, C. Kisielowski, F. H. Baumann, Y. Kim, and A. Ourmazd, *Mapping projected potential, interfacial roughness, and composition in general crystalline solids by quantitative transmission electron microscopy*, Physical Review Letters **71** (1993), 4150–4153.

[SRVR98] P. M. Salzberg, P. I. Rivera-Vega, and A. Rodríguez, *A network flow model for binary tomography on lattices*, Int. J. Imaging Syst. Technol. **9** (1998), 147–154.

[Tes94] D. Tesch, *Disposition von Anruf-Sammeltaxis*, Ph.D. thesis, Universität Passau (1994).

[Tro75] L. E. Trotter, *A class of facet producing graphs for vertex packing polyhedra*, Discrete Math. **12** (1975), 373–388.

[Woe96] G. J. Woeginger, *The Reconstruction of Polyminoes from their Orthogonal Projections*, Technical report, Karl-Franzens-Universität Graz & Technische Universität Graz (1996).

Acknowledgments

A number of persons and circumstances contributed to this work and without their support it would have been impossible to write this thesis.

First of all, I wish to thank my advisor Peter Gritzmann for attracting me to the exciting field of discrete tomography and all his support and cooperation. I am thankful for the employment he provided to me first through the *Graduiertenkolleg Mathematische Optimierung* in Trier, and then through positions with his group in Munich. Furthermore, I am grateful for the independence he allowed me in my research.

Next I want to thank the *Deutsche Forschungsgemeinschaft* for supporting me through a fellowship at the *Graduiertenkolleg Mathematische Optimierung* in Trier and the *Bundesministerium für Bildung und Forschung* for supporting my employment and me through grant 03-GR7TM1 in Munich.

I thank Peter Schwander for inviting me twice to the Institute for Semiconductor Physics in Frankfurt(Oder) where he provided the opportunity for me to learn a lot about electron-microscopy.

I thank my former and current colleagues and friends René Brandenberg, Andreas Brieden, Abhi Dattasharma, Richard J. Gardner, Mariel Lüdecke, Michael Nast, Dieter Prangenberg, Thorsten Theobald, Lionel Thorens, and Markus Wiegelmann for many interesting discussions and the proofreading they all did. Furthermore I thank Jens Zimmermann for helping with coding the data-structures and some of the algorithms utilized in Chapters 3, 4, and 7.

Subsections 2.1.1 and 2.1.2 are based on joint work with P. Schwander. Chapter 3 is based on joint work with P. Gritzmann. The paper [GVW] jointly written and submitted with P. Gritzmann and M. Wiegelmann inspired me to write Chapter 4 that generalizes the theoretical results from [GVW] and adds more empirical evidence for the superior performance of the implemented algorithms. I am thankful to Eddie Cheng (Oakland University) for working with me on matters that are not webs; I report about the joint results in Chapter 5. Chapter 7 is again based on joint work with P. Gritzmann.

Thanks also to Ralf Borndörfer whose thesis motivated me to write Chapter 6. Finally, thanks are due to Pierre A. Stadelmann for permission to use his software package *EMS*.

CHAPTER 1

Introduction

The present thesis studies various problems that are motivated by the demand in material sciences to study the surface of semiconductor chips. A recent, promising technique developed by physicists is able to compute from few electron microscopic images the number of atoms in columns of the crystal in few directions. But that tool cannot reconstruct the spatial location of the atoms in the studied sample. We will present in this thesis methods to reconstruct 3-dimensional configurations that fulfill the given measurements. We will report on computational experiments conducted with these algorithms. These reconstruction algorithms might permit the semiconductor industry to fine-tune the production conditions for chips. Furthermore, problems related to the reconstruction problem of discrete tomography are studied.

In this chapter we give an overview over the area of discrete tomography and put discrete tomography into the broader context of tomography (Section 1.1). Then we explain related problems (Section 1.2). Finally we give a thorough overview of our results (Section 1.3). The relation among all areas is summarized on page xx, Table 0.1.

1.1. (Discrete) Tomography

The basic problem of tomography is to reconstruct an unknown function f that maps some known domain into the set of nonnegative real numbers. About f one of the following is known:

1. the projections of f onto different subspaces,
2. the sums of f over different subspace, or
3. the integrals of f over different subspace.

The classical application of tomography is to obtain a 3-dimensional reconstruction of the unknown density of tissue in a human body from X-rays along several hundred directions. The images of the X-rays correspond to integrals of the unknown density function along lines. Mathematically speaking, the basic problem of this application is to invert a Radon transform (see Radon [RAD17]) if the domain is 2-dimensional or to invert an X-ray transform if the domain's dimension is at least 3; so the methods used for it come mainly from numerical analysis, functional analysis, and in particular Fourier analysis. This method is nowadays a well established technique as anybody can witness who underwent *computerized tomography.* (Of course, it is nevertheless desirable to obtain better reconstructions that reveal more details in shorter time from less data, but this is not our metier.)

Various other classes of tomographic problems result, if the class, the domain, or the range of the functions is restricted. Next we want to introduce two important subclasses.

The first class is obtained by restricting the functions to those that can be described as characteristic functions of geometric subsets of $\mathbb{R}^2$ or $\mathbb{R}^3$. This is the area of *geometric tomography;* for an excellent state-of-the-art survey of geometric tomography we recommend Gardner's book [GAR95]. Usually the class of functions is further restricted to characteristic functions of *convex bodies.* One problem to be mentioned is the question about conditions that guarantee unique reconstruction for a planar convex body. This problem was solved by Giering [GIE63] who showed that for every planar convex body there exist three directions, so that the body is uniquely determined by projections along these three directions.

The second class, called *discrete tomography*, results if the domain and image of the unknown function is restricted to be discrete. Usually, one wants to reconstruct a finite subset of a lattice from the knowledge of the size of intersections of the unknown set with lines of various lattice directions; the size of this intersection is of course just the sum of the characteristic function along the candidate points on this line.

Apparently the term 'discrete tomography' was coined by Larry Shepp who organized, in 1994, a DIMACS Mini-Symposium with this title. Nevertheless, the area of discrete tomography is older than its only recent baptism might suggest. In fact, the early results in discrete tomography were either worded as questions about measurable sets (see, for example Lorentz [LOR49]) or as questions about binary matrices (see, for example Ryser [RYS57] and [RYS63, Chapter 6]) or as questions about finite Radon-transforms (see, for example Bolker [BOL87] who studies invertability for the affine, projective, and k-set transforms).

Later, the uniqueness question for convex sets (that is: *convex lattice sets*) was raised in discrete tomography. It was answered by Gardner and Gritzmann [GG97] who showed that every planar convex lattice set is uniquely determined (among the class of planar convex lattice sets) by X-rays along 4 suitable lattice directions or along any 7 mutually nonparallel lattice directions.

After many years during which questions about structural properties and about algorithms either for two directions (for example, [Rys57, Lor49]) or very many directions (see [Bol87]) were studied, the field gained additional momentum by a new, important application in material sciences.

This primary new application stems from a novel method called QUANTITEM (by Schwander, Kisielowski, Baumann, Kim, and Ourmazd [SKB+93]; and Kisielowski, Schwander, Baumann, Seibt, Kim, and Ourmazd [KSB+95]) for analyzing images obtained by high resolution transmission electron microscopy. QUANTITEM permits (under certain circumstances) to *count* atoms in each atomic column of a crystal. This method is particularly well suited to look through the amorphous layer on top of silicon wafers. The ability to analyze the hidden surface of the wafer would be very helpful for the semiconductor industry to further fine-tune the production parameters. But the underlying type of microscope and sample presents new challenges:

1. measurements can be performed only along up to 5 directions, because the damage to the specimen caused by the high energy electron beam in the microscope is negligible only for up to 5 directions; for more directions the damage becomes more serious;
2. the microscope permits only a restricted tilt-angle;
3. the analysis by QUANTITEM and the resolution of the used detector (CCD) permit only measurements along zone-axes of low index;
4. the specimen may not be convex.

Recently the emphasis of research has shifted more towards algorithmic questions related to discrete tomography. There are some articles that prove that certain tasks are computational intractable (see Gardner, Gritzmann, and Prangenberg [GGP99]; Woeginger [Woe96]; Gritzmann, Prangenberg, de Vries, and Wiegelmann [GPVW98]; and Barcucci, Del Lungo, Nivat, and Pinzani [BLNP96]). Then, there are reports on computationally tractable tasks like the reconstruction for two directions, see Anstee [Ans83]. Also, one should mention several experimental studies like Salzberg, Rivera-Vega, and Rodríguez [SRVR98]; and Matej, Herman, and Vardi [MHV98].

Many results of this dissertation lie in this more algorithmically flavored area of discrete tomography.

1.2. Packing, Covering, and Stable Sets

A natural way to look at problems of discrete tomography is in terms of an interpretation with query sets. Each line in a given instance is a query set and the measurement along this line describes how many elements of this set should be chosen. This problem constitutes a generalized set partitioning problem. As the reconstruction problem in the important case of at least 3 directions is $\mathbb{NP}$-complete (see [GGP99]) it is natural to consider relaxations, where the numbers associated with the query sets describe only upper bounds (or lower bounds) on the number of elements and one seeks a solution with as many (as few) elements as possible. These two relaxations have the form of a generalized set packing problem and of a generalized set covering problem, respectively.

The class of set partitioning problems is used most frequently in literature to solve airline crew scheduling problems (see for a recent article, Hoffman and Padberg [HP93]), and Dial-a-Ride problems of different types (see Tesch [Tes94] and Borndörfer [Bor97]).

Partitioning problems provide a powerful language to model these applications easily. But the algorithms to solve concrete instances usually exploit the fact that a solution of a partitioning problem can equivalently well be described as a simultaneous solution of a packing and a covering problem. This is the reason why for all applications it is very important to study packing and covering problems on their own. In Chapter 4 we present new results for obtaining approximate solutions of generalized cardinality set packing and covering problems.

A special (though not that special) case of set packing problems is constituted by stable set problems on graphs. Here the task is to find the largest subset of vertices of a graph, so that all vertices of this subset are nonadjacent. This constitutes a packing problem when the edges are considered as query sets of capacity 1.

It might appear to be more surprising, that every set packing problem can be transformed into a stable set problem. This transformation is done by interpreting all the query sets (that have in this case capacity 1) of the set packing problem as cliques of the stable set problem.

1.3. Overview and Main Results

The main part and contribution of this dissertation are Chapters 3–7. In the present section we give an overview of the results obtained in the different chapters. The chapters are mainly self-contained except for the major part of the common, preliminary material, which is presented in Chapter 2.

Chapter 2: We provide a thorough introduction into the physics underlying discrete tomography and explain how images are formed in a high resolution transmission electron microscope. Then, we sketch how it is possible to recover the number of atoms on lines by QUANTITEM. Finally we report on new results we obtained by first simulating the electron microscope on a tiny sample and then analyzing the results with an implementation of QUANTITEM to recover the number of atoms on the atom columns. It turns out that our results (up to noise) nicely match the theoretical predictions. Thereby we provide additional evidence that it is indeed possible to recover the atom-counts from an image obtained by high resolution transmission electron microscopy. Even though the microscope has two axes of freedom, one of them is used entirely to maintain the field of vision; so only one degree of freedom can be used for different images. Hence all measurement directions lie in a single plane and the (originally 3-dimensional) reconstruction problem decomposes quite naturally into a set of independent, 2-dimensional problems. Finally, we state the most important mathematical problems related to the reconstruction of the spatial data from few microscopic images.

Chapter 3: The machinery of polyhedral combinatorics is used as a mighty tool to formulate algorithmic problems of discrete tomography like *reconstruction, uniqueness*, and *invariance* in a unifying framework. First the *tomography polytope* is defined and then two new classes of valid inequalities are introduced and conditions are studied that guarantee facetness for the induced faces.

We investigate algorithms to solve the reconstruction problem for 3 directions by integer programming and report then about computational results of an implementation. Problems of size 70×70 were on average solved within 9 minutes; even problems of size 100×100 that involve 10000 variables can be solved. Given these encouraging results we propose an algorithm which provides in any given step either a solution, or a good lower bound. With the help of this lower bound we can then compute a valid cut that is violated by the current fractional solution. The computation of the cut involves solving a tiny instance of an $\mathbb{NP}$-hard problem.

Chapter 4: Even though the methods introduced in Chapter 3 suffice to solve problems of size up to 100×100 there is demand to solve even larger problems. But for the time being it appears infeasible to solve problems of sizes up to 500×500 exactly. Hence we investigate relaxations of the reconstruction problem. It turns out, that these relaxations have the form of generalized cardinality packing and covering problems. For both problems we propose simple greedy-type and more elaborate improvement-type approximation algorithms. We obtain bounds for their performance, which are in the case of the packing problems sharp.

In practice, all tomographic problems are dense. Hence we look into the generalized packing and covering problems for dense instances. For them we obtain a *polynomial time approximation scheme*. This result should be regarded as a primarily theoretical result, as the involved 'constants' are intractably large.

Finally we specialize the approximation algorithms to discrete tomography. The resulting theoretical bounds are new, as are the algorithms. We provide some examples indicating that the theoretical bounds are sharp. Then we report about our quite successful implementation of the packing algorithms for discrete tomography. In practice, they outperform the theoretical bounds by a large factor. The best of our algorithms usually manages to fill in more than 99% of the required atoms.

Chapter 5: After we considered in the previous chapter a generalization of discrete tomography in this chapter we investigate as a particular application of set packing problems the stable set problems. They have been thoroughly studied in their own right. But today they are studied more for their property that they constitute one of two relaxations of the set partitioning problems, as stable set problems and standard set packing problems are equivalent. (Of course, many problems are equivalent in being $\mathbb{NP}$-complete but stable set problem and set packing are just two different sides of a single coin.)

The polyhedral approach is frequently preferred for set partitioning problems because it permits easily to incorporate and exploit knowledge about both relaxations simultaneously. Here, we study the polyhedral description of the stable set problem. Classes of facet defining or valid inequalities are most helpful if their separation problem can be solved. The separation problem is to find a violated inequality in this class for a given fractional solution. Therefore, we place special emphasis on the separation problem. For the class of antiweb inequalities defined by L. E. Trotter Jr. [Tro75] we answer in particular the following three questions:

1. What can be said about the separation problem for antiweb inequalities?
2. How *strong* are antiweb inequalities?
3. What common generalization do the classes of antiweb inequalities and of generalized wheel inequalities permit?

It is very surprising that for the separation problem of antiweb inequalities, which solution is very important to use them in practice, nothing new was discovered since their introduction in 1975. This is even more astonishing, as they encompass the odd cycle and clique inequalities as different extreme cases. For odd cycle inequalities a separation method is due to Grötschel, Lovász, and Schrijver [GLS93]; for the clique inequalities the separation problem is $\mathbb{NP}$-complete, though they are contained in the polynomially separable class of *orthonormal representation cuts* [GLS93, 9.3.2]. To our knowledge, antiweb inequalities have never been incorporated as valid cuts into any integer programming solver. An important feature of antiweb inequalities is, however, that for any $k \geq 2$ they encompass inequalities whose support graph is k-connected; thereby they can utilize higher connectivity of problem instances. On the other hand, the support graphs of the frequently used odd cycle inequalities are just 2-connected; this is at the same time just the minimum requirement for support graphs of any facet; here antiweb inequalities have their special strength. We prove that—though their separation problem is $\mathbb{NP}$-complete in general—there is a natural sequence of inequality-classes that *are* separable in polynomial time; at the same time they are of higher connectivity than just 2-connected and their (infinite) union contains *all* antiweb inequalities.

Further, we prove lower bounds for the *Lovász-Theta function* [Lov79] of antiwebs hence showing that the antiweb inequalities are not implied by the class of orthonormal representation cuts, which are not combinatorially defined.

In addition we provide a unifying framework for the concepts of generalized wheel inequalities (see Cheng and Cunningham [CC97]) and antiweb inequalities. We provide a common generalization—the new class of antiweb-wheel inequalities. *Algorithms* are provided to separate the members of a corresponding hierarchy in *polynomial time*. Finally all facet inducing inequalities among the proper antiweb wheel inequalities are *characterized* by utilizing graph compositions that maintain facetness.

Chapter 6: After studying different issues related to antiwebs in the previous chapter we continue now with a look at the stable set polytope from a different angle. We observed that at times it is quite tedious to prove for new classes of inequalities that they are facet inducing. To alleviate this burden different operations are defined in the literature that permit to reduce the proof of facetness of complicated configurations to a proof for simpler configurations. Examples to mention in this direction are Chvátal's [CHV75] substitution of a graph into a vertex of another graph and Cunningham's [CUN82] composition. Stimulated by some new techniques by Borndörfer and Weismantel [BW97] (see also [BOR97, Chap. 2]) we introduce the new graph operation of *partial substitution* that generalizes Chvátal's and Cunningham's operations. We provide conditions that guarantee that the resulting inequality after partial substitution of two facets yields a facet again. Generalizing an idea of [BW97] we give then a polynomial time separation algorithm for a class of inequalities constructed by means of partial substitution.

Chapter 7: Finally we compare two different models for the reconstruction problem of discrete tomography that have been proposed in the literature. In the first the only requirement is that the configuration satisfies the measurements. In the other model, in addition, convexity along the lines of one direction is required. We employ exact solution algorithms that compute for a given configuration of atoms the most different other configuration (with respect to symmetric difference) that is not distinguishable via its given X-rays. This method permits to evaluate directly the information content of the different models. Even though it is clear that the model with *line-convexity* should permit less nonuniqueness than the unrestricted model it is still surprising how large the nonuniqueness is for the unrestricted problem. Entirely different is the situation for the model with line-convexity for instance with high density where the worst possible symmetric differences we could observe were on average a lot smaller than in the general (not line-convex) case. It is still not completely clear yet, whether the assumption of line-convexity applies to the microscopic applications.

Finally—for the reader's convenience—we provide a table showing the interaction among the different parts. In Table 1.1 the rows are labeled with the different applications (Discrete Tomography; Packing and Covering; and Stable Set Problems); the columns are labeled with the solution approach (either via Polytopes or Algorithms).

	Polytopes	Algorithms
Discrete Tomography	3	2.5, 3.6, 4.6, 7
Packing and Covering	3.2	4
Stable Set Problems	5, 6	4.5, 5.3–5.7, 6.3

Table 1.1. Relation of the different parts of this dissertation to the covered topics.

CHAPTER 2

Preliminaries

The purpose of this chapter is to introduce the most important concepts used later. Of course, this cannot be an in-depth description covering every smallest detail. For further details on the involved topics we will give pointers to the literature.

2.1. Discrete Tomography

In this section we briefly describe high resolution transmission electron microscopy (abbreviated by *HRTEM*). Then we will report on some microscope simulations we performed[1] on tiny silicon wedges to create realistic (though small) phantoms. Finally, we present the analysis of these phantoms by the quantitative analysis of the information provided by transmission electron microscopy (abbreviated by *QUANTITEM*) by Schwander, Kisielowski, Baumann, Kim, and Ourmazd [SKB+93] and Kisielowski, Schwander, Baumann, Seibt, Kim, and Ourmazd [KSB+95] in order to recover the number of atoms on the atomic columns.

2.1.1. High Resolution Transmission Electron Microscopy. In the early 1930's conventional light-microscopy was pushed to its theoretical limits to provide a resolution of ca. 0.5 μm. Due to the demand for even better resolutions people tried to replace the light beam of traditional microscopy (whose relatively long wavelength caused the earlier mentioned limits) by an electron beam (of shorter wavelength). Ernst Ruska (under the supervision of Max Knoll) was in 1931/32 the first to build a machine utilizing an electron beam to look "through" a specimen. In 1986 he was awarded the *Nobel Prize in Physics* for this outstanding accomplishment (and fundamental work in electron optics). The first transmission

[1] jointly with P. Schwander, Institute for Semiconductor Physics, Walter-Korsing-Str. 2, 15230 Frankfurt (Oder), Email: `schwander@ihp-ffo.de`

electron microscopes *(TEM)* were mainly used to study biological specimens. These TEM's "almost" work like the light microscopes "except" that the light beam is replaced by an electron beam and the optical elements are replaced by electron-optical analogues (e.g., lenses of glass are replaced by electromagnetic or electrostatic lenses). Images are created in a TEM by recording the electron beam intensities after it passed through the specimen; areas of it with heavier atoms or greater thickness diffract more electrons away from the detector; so they permit less electrons to pass through. For TEM the picture depends strongly on the variation of thickness and atomic number in the specimen.

For very thin specimen the limits of available resolution were pushed even further down to atomic resolution with the introduction of high resolution transmission electron microscopes *(HRTEM).* Almost no absorption occurs for thin samples in a HRTEM with a high acceleration potential (circa 200 keV). So image formation is influenced by other principles. As there is no loss of energy the resulting wave function Ψ of the interaction between the electron beam and the electrostatic potential of the crystal is governed by the time independent Schrödinger equation:

$$\nabla^2\Psi(r) + \frac{8\pi^2 me}{\hbar^2}[E + V(r)]\Psi(r) = 0. \tag{2.1}$$

Here, e is the electronic charge, E the acceleration potential of the microscope, $\hbar$ is Plank's constant, m is the mass of the electron, and $V(r)$ is the crystal potential at position r. This creates an electron wave emanating from the "bottom side" of the crystal (if the electron beam hits the specimen from "above"). It is this wave that is then magnified by the electrostatic lens to obtain the final image. Unfortunately, at this high magnification the lens distorts the image rather strongly; to worsen matters, this distortion depends very sensitively on the imaging condition (weather, horoscope, etc.) that cannot be fully determined. So traditionally, given an HRTEM image the art is to do simulations of different objects under different imaging conditions until a simulated phantom was created that matched the obtained image. This explains the importance of simulation. In the next subsection we will give an overview about one particular simulation method: the *Multislice Method* by Cowley and Moodie [CM57].

2.1.2. Simulations and Analysis. The objectives of this subsection are twofold. First we want to describe how the electron wave at the exit face of a specimen can be calculated from the specimen's crystal potential and the imaging conditions using the multislice method. Then we will show

how to analyze the (by the lense distorted) image, to recover the height information using QUANTITEM.

The main issue with the simulation is to solve the Schrödinger equation to obtain a description of the wave at the exit face of the specimen. To do the simulation it is important to know that the involved energies are of quite different orders of magnitude: on one side the low energy potential field of the crystal and on the other side the electron beam of about 200 keV. These different scales lead to a situation, where most of the electrons of the beam are scattered only forward (there is almost no backscattering) and the scattering occurs only by a very small angle.

So under these approximations the simulation problem is more a problem of wave propagation. This permits to do a slice-wise simulation, by cutting the specimen into thin slices orthogonal to the beam's direction. The whole potential of the slice is projected onto the plane of the slice closer to the electron beam's source (entry face). It turns out that the interaction of the beam with the projected potential on the entry face can be described (under the given approximations) by a phase shift for the beam's wave. Mathematically speaking, the phase shift is done by multiplying the wave function by $\exp(i\phi(r))$ where $\phi(r)$ is the phase shift incurred at location r. The propagation of the resulting wave from one entry face to the next is then done by using the Fresnel approximation.

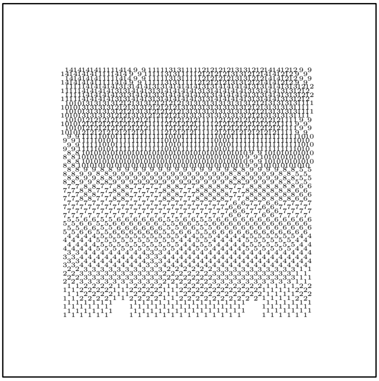
The leaving wave is modeled as the sum of many spherical (actually, in this approximation, paraboloidal) waves. Their joint effect on the next entry plane is modeled by a convolution integral. Following the—then novel—physical optics approach of Cowley and Moodie [CM58] (as done in [CM57]) it turns out that the process is best described, by multiplying the incoming wavefront with the phase shift exponential; the propagation to the next layer is then facilitated by a convolution. Ishizuka and Uyeda showed in [IU77] that the result of this computation is actually an approximate solution to the Schrödinger equation (2.1). Furthermore, Goodman and Moodie [GM74] show that in the case where the slice thickness goes to 0, the simulation results converge to a solution of a modification of the Schrödinger equation (2.1) that excludes backscattering—an assumption we already agreed on.

Computationally, the convolution is better performed by using the fast Fourier transform (first introduced by Good [Goo58] and later by Cooley and Tukey [CT65] for power of two orders; for orders with other factorization see for example de Boor [Boo89]) to transform the phase shifted wave function to Fourier space, the convolution reduces to a simple multiplication there. Then the result is inversely Fourier transformed back into the real space. This process is done for all slices. Finally the magnifying electrostatic lens is simulated by a similar procedure. For the simulation of the HRTEM we used the package *EMS* by Stadelmann [Sta87]. As specimen we always used a wedge of silicon, which is of face-centered cubic *(fcc)* crystal type (for more on crystallography see for example Borchardt-Ott [BO97]), embedded into a cube of $15 \times 15 \times 15$ unit cells. The topmost unit cells are removed at random. The resulting atomic columns viewed at $\langle 001 \rangle$ (from above) had varying heights between 0 atoms and 14 atoms.

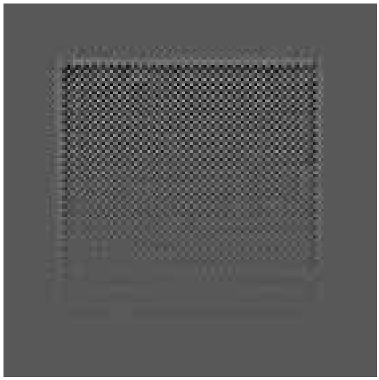
From this wedge (its height-field is depicted in Figures 2.1(a), 2.2(a), and 2.3(a)), with the help of EMS, we then computed the simulated image for the resolutions 512×512, 2048×2048, and 4096×4096 subdivided into 30 slices in each case. The simulated images are given in Figures 2.1(b), 2.2(b), and 2.3(b). It turned out that the (later given) analysis for the phantoms with resolution 2048×2048 and 4096×4096 was the same, thereby indicating a certain degree of convergence. So we can next turn our attention to the question of analysis of these pictures.

In [KSB$^+$95] it is shown that even though the image gets distorted by the lens some structural properties are preserved. It turns out that for important specimen examples like Silicon in direction $\langle 100 \rangle$ and Germanium in direction $\langle 100 \rangle$ and $\langle 110 \rangle$ the wave function at the exit face of the specimen can be well approximated by the superposition of only two Bloch waves (fundamental solutions of the underlying PDE). The coordinates (with respect to the intensity of two excited Bloch waves) of the intensities of all solutions to the PDE that fulfill the necessary boundary and initial value conditions all lie on an ellipse. Furthermore, the angle (to a fixed reference point on the ellipse's rim) corresponds to the height of the sample at that place. This all holds for lattice directions and the two beam case. In the case of lattice directions an atomic column really influences only a small corresponding area—the height information is localized. [KSB$^+$95] observe that the image maintains this property after the lens magnified and distorted it; that is, images can still be composed of two basis images (approximately).

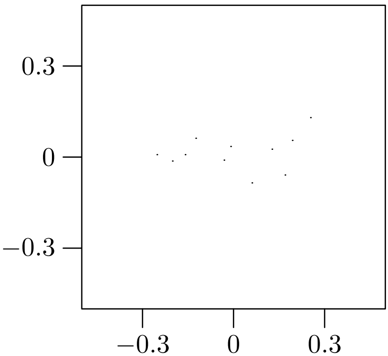
In [KSB$^+$95] this insight is utilized by segmenting the picture into small rectangles that are mostly influenced by only a single atomic column. Each of these rectangles is considered an image vector. By translating them, the center of gravity of all of these vectors is moved to the origin. Next the covariance matrix of the image vectors is computed and for it a principal



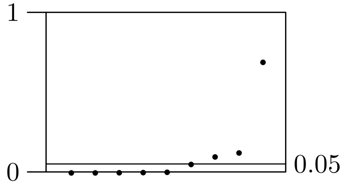
(a) Height of the wedge.



(b) HRTEM simulation.

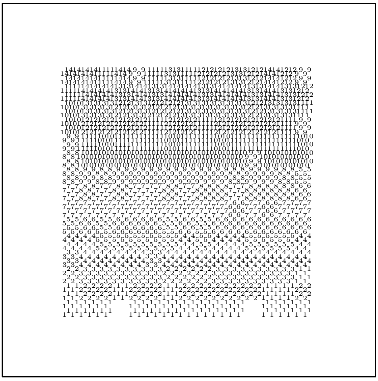


(c) Coordinates of the image-cells with respect to the largest and second largest eigenimage.

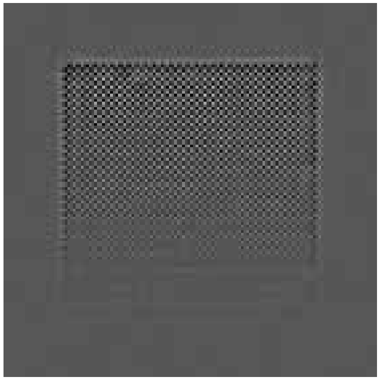


(d) Weights of the largest eigenvalues.

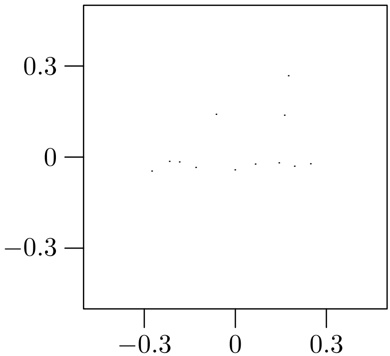
FIGURE 2.1. HRTEM-simulation of a $15 \times 15 \times 15$-wedge at resolution 512×512. The tiny(!) numbers in (a) show the heights of the corresponding columns.



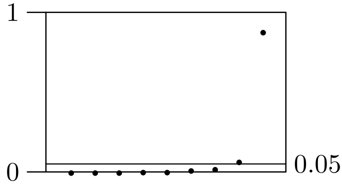
(a) Height of the wedge.



(b) HRTEM simulation.

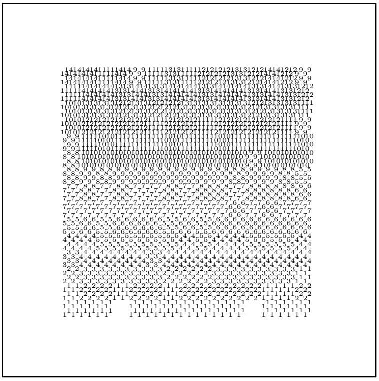


(c) Coordinates of the image-cells with respect to the largest and second largest eigenimage.

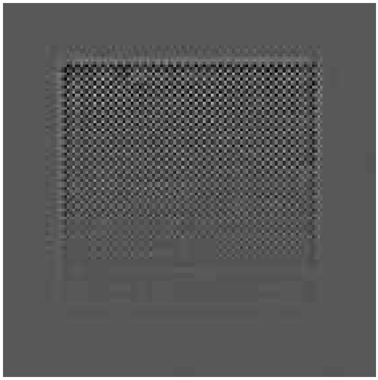


(d) Weights of the largest eigenvalues.

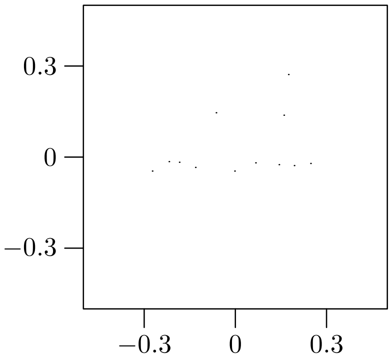
FIGURE 2.2. HRTEM-simulation of a $15 \times 15 \times 15$-wedge at resolution 2048×2048.



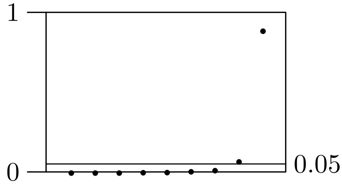
(a) Height of the wedge.



(b) HRTEM Simulation.



(c) Coordinates of the image-cells with respect to the largest and second largest eigenimage.



(d) Weights of the largest eigenvalues.

FIGURE 2.3. HRTEM-simulation of a $15 \times 15 \times 15$-wedge at resolution 4096×4096.

component analysis is done. The image vectors of the two most influential principal components correspond now to the intensities of the magnified (and distorted) two Bloch waves, so in theory it suffices to express all other image vectors as linear combinations of these two and then to read off the angle on the ellipse to infer the height. Up to now we have only said, that the angle of the ellipse is related to the height of the column. In [KSB+95] two circumstances are described among others, that permit to make this relation fully precise:

1. if the sample thickness is distributed uniformly, then the local density of the image vectors on the ellipse rim is inversely proportional to the local rate of path traversal on the rim;
2. if there is very little noise, then the discrete nature of the different heights leads to a clustering of imagevectors on the ellipse rim corresponding to the different heights.

For our experiments we cut out a rhombic region around the centers of the columns (tiles of the Voronoi diagram of the projection; if in the projection dumbbells occur then they are first contracted) and then used the pixels inside it to form the image vectors. Then we computed the principal components for the resolutions 512×512, 2048×2048 and 4096×4096 and plotted their projections onto the two eigenimages with the largest eigenvalues in Figures 2.1(c), 2.2(c), and 2.3(c), respectively. The quality of the simulation and analysis can be inferred from the values of the largest eigenvalues. Theory says, that there should be two significant eigenvalues and all others should be smaller than 0.05, see Figures 2.1(d), 2.2(d), and 2.3(d). For the lowest resolution, the picture violates this; but for the two better resolutions this criterion is fulfilled. Actually, one would expect both the largest two eigenvalues to be similar in magnitude; but as the wedges we study are even thinner than the theory requires, we see only a small piece of the ellipse rim and that looks locally like a line; so one eigenvalue should (and does!) dominate the other one.

The projection onto the main eigenimages in Figure 2.1(c) reveals only little information for the lowest resolution, as the incurred errors are too large. But in Figures 2.2(c) and 2.3(c) there is clearly a motion from the left (corresponding to columns of height 0) to the right (corresponding to columns of height 14). The vertical expansion is mainly due to errors at the boundary of the sample and the fact that the sample is still very thin compared to the extinction length that lies somewhere between 150 and 200 atoms. Because there is almost no change between the simulations with 2048×2048 and 4096×4096 the simulation should be stable now.

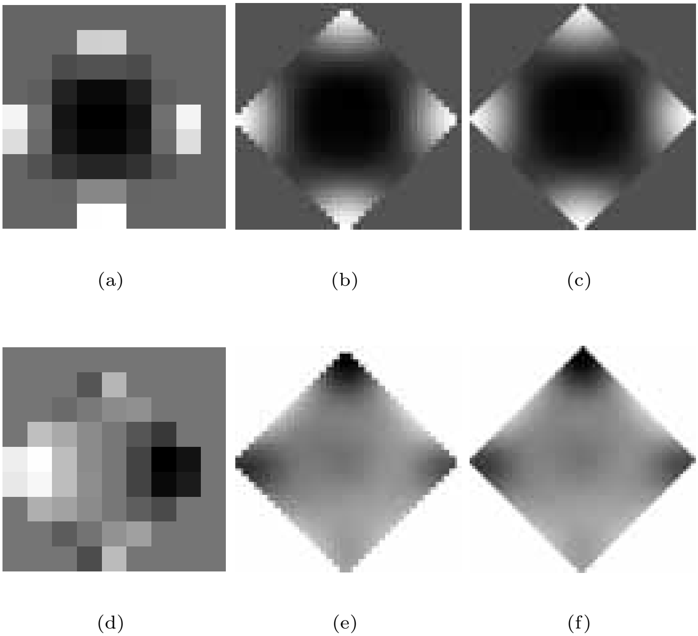


Figure 2.4. Comparison of the most *(top)* and second most *(bottom)* dominating eigenimages for HRTEM-simulations of a $15\times15\times15$-wedge at resolution 512×512, 2048×2048, and 4096×4096 *(from left to right)*.

This impression is also verified by the large similarities between the two eigenimages at medium resolution (2.4(b), 2.4(e)) and those at high resolution (2.4(c), 2.4(f)), respectively. Even though the second most important eigenimages 2.4(e), 2.4(f) are of very little influence here, the slight asymmetry between their top and bottom part can be nicely explained by the fact that the bottom-top direction is the direction along which the wedge rises; so there should be some asymmetry.

After we established the quality of the two better simulations it turns out that the dots in the projection image onto the two eigenimages with the largest eigenvalues in 2.2(c) and 2.3(c) really show a nice clustering. Counting the clusters easily reveals 14 clusters; only the clusters corresponding to columns of height 0 and 1 cannot be separated. But this demonstrates, in principle, that with the outlined techniques it should be possible to count atoms in different columns (perhaps up to an error of ± 1).

2.1.3. Basics of Discrete Tomography. We use the general setting of a d-dimensional Euclidean space $\mathcal{E}^d$, with $d \geq 2$, though only the cases $d = 2, 3$ are relevant for electron microscopic applications. Let $\mathcal{S}_{1,d}$ be the set of all 1-dimensional subspaces in $\mathcal{E}^d$, and let $\mathcal{F}^d$ denote the family of finite subsets of $\mathbb{Z}^d$. For $F \in \mathcal{F}^d$ let $|F|$ be the cardinality of F. A vector $v \in \mathbb{Z}^d \setminus \{0\}$ is called a *lattice direction*; $\mathcal{L}_{1,d}$ denotes the subset of $\mathcal{S}_{1,d}$ spanned by a lattice direction. For $S \in \mathcal{S}_{1,d}$ let $\mathcal{A}(S)$ denote the family of all lines parallel to S. The *(discrete)* 1*-dimensional X-ray parallel to* S of a set $F \in \mathcal{F}^d$ is the function $X_S F : \mathcal{A}(S) \to \mathbb{N}_0 = \mathbb{N} \cup \{0\}$ defined by

$$X_S F(T) = |F \cap T|, \qquad \text{for } T \in \mathcal{A}(S).$$

Since F is finite, the X-ray $X_S F$ has finite support $\mathcal{T} \subset \mathcal{A}(S)$.

Actually, in practice the microscope is tilted only in a single plane; therefore, all the measurement directions lie in this plane and the (originally 3-dimensional) problem decomposes naturally into a set of independent, 2-dimensional problems. So henceforth we will mainly study 2-dimensional problems.

In the inverse reconstruction problem, we are given *candidate functions* $\phi_i : \mathcal{A}(S_i) \to \mathbb{N}_0$, $i = 1, \dots, m$, with finite support and want to find a set $F \subset \mathbb{Z}^d$ with corresponding X-rays. More formally, for $S_1, \dots, S_m \in \mathcal{L}_{1,d}$ pairwise different, the most important algorithmic task in our context can be stated as the following search problem:

RECONSTRUCTION $(S_1, \dots, S_m)$.

Instance:	*Candidate functions* $\phi_i : \mathcal{A}(S_i) \to \mathbb{N}_0$ *for* $i = 1, \dots, m$.
Output:	*If the instance is consistent, a finite set* $F \subset \mathbb{Z}^d$ *such that* $\phi_i = X_{S_i} F$ *for all* $i = 1, \dots, m$; *otherwise the answer* no.

Clearly, when investigating the computational complexity of the preceding problem in the usual *binary Turing machine model* one has to describe suitable finite data structures. We do not go into such details here but refer the reader to Gardner, Gritzmann, and Prangenberg [GGP99]. For the purpose of this dissertation, handling an input of m candidate functions $\phi_1, \dots, \phi_m$ with supports $\mathcal{T}_1, \dots, \mathcal{T}_m$, respectively, is facilitated with the aid of a set $G \subset \mathbb{Z}^d$ of candidate points. This set G consists of the intersection of all (finitely many) translates of $\bigcap_{i=1}^m S_i$ that arise as the intersection of m lines parallel to $S_1, \dots, S_m$ with $\mathbb{Z}^d$, respectively, whose candidate function value is nonzero, i.e.

$$G = \mathbb{Z}^d \cap \bigcap_{i=1}^{m} \bigcup_{T \in \mathcal{T}_i} T.$$

To exclude trivial cases we will in the following always assume that $G \neq \emptyset$ and that $\bigcap_{i=1}^m S_i = \{0\}$. Hence, in particular $m \geq 2$.

The incidences of G and $\mathcal{T}_i$ can be encoded by an *X-ray–candidate-point incidence matrix* A_i. To fix the notation, let G consist of, say, N points, let $M_i = |\mathcal{T}_i|$ and $M = M_1 + \cdots + M_m$. Then the incidence matrices $A_i \in \{0,1\}^{M_i \times N}$ can be joined together to form a matrix $A \in \{0,1\}^{M \times N}$. Identifying a subset of G with its characteristic vector $x \in \{0,1\}^N$, the reconstruction problem amounts to solving the integer linear feasibility program

$$Ax = b, \quad \text{s.t.} \quad x \in \{0,1\}^N, \tag{2.2}$$

where $b^T = (b_1^T, \dots, b_m^T)$ contains the corresponding values of the candidate functions $\phi_1, \dots, \phi_m$ as the right hand sides of $A_1, \dots, A_m$, respectively.

For $m = 2$ the matrix A is totally unimodular and this permits already a polynomial time solution, see Section 2.5. For $m = 3$ the system $Ax = b$ has the form of a planar 3-dimensional transportation problem; for more on transportation problems see Emeličev, Kovalev, and Kravcov [EKK85, Chapters 6–8].

Let us point out here in passing that more general inverse discrete problems can be modeled in a similar way. In fact, query sets (which are just lines in the case of the discrete tomography considered here) could be chosen in various different and meaningful ways. (For instance, if the lines are replaced by the translates of some k-dimensional subspaces we obtain the reconstruction problem for discrete *k-dimensional X-rays*.) For example, Chapter 4 is phrased in this more general language of *query sets*.

For $m \leq 2$ it is well-known that Reconstruction $(S_1, \dots, S_m)$ can be solved in polynomial time, see Section 2.5 for a list of different reasons.

However, RECONSTRUCTION $(S_1, \ldots, S_m)$ becomes $\mathbb{NP}$-hard, for $m \geq 3$, [GGP99]. This means, that (unless $\mathbb{P} = \mathbb{NP}$) exact solutions of (2.2) require (in general) a superpolynomial amount of time.

Let us stress the fact that while the solutions of the polynomial-time solvable LP-relaxation of (2.2) does provide some information about (2.2) (see [FSSV97], and in this thesis Sections 3.4 and 4.4), it is the goal to solve (2.2) rather than its LP-relaxation, since the objects underlying our prime application are crystalline structures forming (physical) sets of atoms rather than 'fuzzy' sets; for some additional discussion of this point see [GPVW98].

2.2. Computational Complexity

In this section we give an overview of the theory of computational complexity. This will not be an in-depth description but we intend to define the terms used in Chapter 4 and Section 5.4. For further details on computational complexity see the books by Garey and Johnson [GJ79] and by Papadimitriou [PAP94].

First we need a concept to compare running times (frequently also measured by the number of primitive instructions it takes to solve a problem) of different algorithms. This concept should be able to distinguish among different orders of magnitude of growth (e.g., n, and n^2, and 2^n should be different by this measure). But as CPU clock-speeds of computers vary strongly, this concept should not distinguish between n^2 and $4n^2$. A side effect of this requirement—almost everybody will agree, that between n^2 and $4n^2$ there is only a small difference—is that, to make this an equivalence relation, transitivity requires also to regard n^2 and $10^{100}n^2$ as equivalent. The *big-Oh* notation is used to measure complexity. For two functions $f, g : \mathbb{N} \longmapsto \mathbb{N}$ one says that $f(n) = O(g(n))$ if there exists a $c > 0$ such that for almost all $n \in \mathbb{N}$ holds $f(n) \leq c \cdot g(n)$.

Computational complexity is concerned with measuring the difficulty of various problems and the efficiency of algorithms to solve them. Mathematically speaking, an algorithm *solves* a problem Π if it provides for every possible input *instance* I a solution. Most important for our work here are three particular classes of problems: *decision problems, search problems,* and *optimization problems.*

For a decision problem only the answers *yes* and *no* are permitted, while in a search problem either a certain configuration or the answer *no* is expected. Instances with answer *yes* are called *yes*-instances. For optimization problems the answer is a configuration with a "best possible" value. To measure the difficulty of a particular instance I we consider a reasonable binary encoding. The length of this encoding (that is the number of 0's and 1's) is then the *encoding length* of this instance, denoted by $|I|$. As the model of computation we use that of a deterministic Turing machine (DTM).

Next we seek a coarser way to distinguish complexities of different decision problems. On one side we want to put the problems that have algorithms that need only time polynomial in the encoding length, on the other side we want to put the problems that appear to be more difficult. Algorithms that run in polynomial time on a DTM are called *polynomial*. We say that a problem Π belongs to $\mathbb{P}$ if there is a polynomial time algorithm for Π. We use the partial order $\propto$ to compare the complexities of different problems. A *polynomial transformation* of instances of a problem Π to instances of a problems Π' is a polynomial algorithm that maps instances of Π to instances of Π'. We say for two decision problems that $\Pi \propto \Pi'$ if there is a polynomial transformation f from Π to Π' such that $f(I)$ (for an instance I of Π) is a *yes*-instance of Π' if and only if I is a *yes*-instance of Π. For $\Pi \propto \Pi'$ we say that Π' is at least as difficult as Π, because a polynomial time algorithm for the solution of Π' together with the polynomial transformation from Π to Π' would yield also a polynomial time algorithm for Π.

Next one wonders of course whether all decision problems belong to $\mathbb{P}$; the answer is unknown. But to obtain an idea what might be beyond $\mathbb{P}$ in 1971 Cook (though he uses a slightly different notion of $\propto$) introduced the class $\mathbb{NP}$. A problem Π belongs to $\mathbb{NP}$ if there exists a polynomial p_Π so that for every *yes*-instance I of Π there exists a proof of the fact that I is a *yes*-instance of encodings length at most $p_\Pi(|I|)$. Similarly, a problem Π belongs to co-$\mathbb{NP}$ if there exists a polynomial q_Π so that for every *no*-instance I of Π there exists a proof of the fact that I is a *no*-instance of encodings length at most $q_\Pi(|I|)$. The underlying computational model is here the model of a nondeterministic Turing machine (NTM). Obviously, every problem in $\mathbb{P}$ belongs to $\mathbb{NP}$.

Cook succeeded in proving for a first problem (that of *satisfiability*) that it is among all problems in $\mathbb{NP}$ the "most difficult" problem, therefore he called it $\mathbb{NP}$-*complete*. In the following year Karp [Kar72] utilized Cook's theorem to prove for a list of 20 more problems, that they are $\mathbb{NP}$-complete. Among them is the problem Clique whose $\mathbb{NP}$-completeness we will use later in Section 5.4.

CLIQUE.

Instance: *A graph G and a positive integer k.*

Question: *Does G have a set of k mutually adjacent vertices?*

It is still an open question whether $\mathbb{P} = \mathbb{NP}$ or not.

Then there is the class of $\mathbb{NP}$-*hard* problems. Informally speaking, an optimization or search problem is called $\mathbb{NP}$-hard, if the existence of a polynomial algorithm for its solution would imply that all problems in $\mathbb{NP}$ are solvable in polynomial time.

Finally, we need to speak about optimization problems and in particular approximation algorithms; for a more detailed description see Crescenzi and Kann [CK98]. For an instance I of an optimization problem Π and an approximation algorithm A (an algorithm that has as output a feasible though possibly non-optimal solution) we denote by $\mathrm{OPT}(I)$ the optimal value and by $\mathrm{A}(I)$ the value obtained by algorithm A. For every instance I the performance $R_{\mathrm{A}}(I)$ is defined by $R_{\mathrm{A}}(I) = \mathrm{A}(I)/\mathrm{OPT}(I)$. The *absolute performance ratio* R_{A} is defined as the supremum of all numbers that are joint lower bounds of $R_{\mathrm{A}}(I)$ for all I if Π is a maximization problem and defined as the infimum of all numbers that are joint upper bounds of $R_{\mathrm{A}}(I)$ for all I if Π is a minimization problem.

2.3. Polytopes, Linear and Integer Programming

For a finite set $X \subset \mathbb{R}^n$ with $X = \{x^1, x^2, \ldots, x^m\}$ the *convex hull* $\mathrm{conv}(X)$ is defined by $\mathrm{conv}(X) = \{\lambda^T(x^1, x^2, \ldots, x^m) \colon \lambda \in \mathbb{R}^n, \lambda \geq \mathbf{0} \text{ and } \lambda^T\mathbf{1} = 1\}$. A set P that is the convex hull of finitely many vectors is called a *polytope*. It is well-known that every polytope can equivalently be described as a bounded region given as the finite intersection of halfspaces (see [ZIE95, Thm. 2.15]). Hence for every polytope P in $\mathbb{R}^n$ there exists a $k \in \mathbb{N}$, a $k \times n$ matrix A and a k-vector b so that $P = \{x \in \mathbb{R}^n \colon Ax \leq b\}$. An inequality $\pi^T x \leq \beta$ is *valid* for a polytope P if $x \in P$ implies $\pi^T x \leq \beta$. A hyperplane $H = \{x \in \mathbb{R}^n \colon \pi^T x = \beta\}$ is called *supporting* for a polytope P if either $\pi^T x \leq \beta$ or $\pi^T x \geq \beta$ is valid for P and $H \cap P \neq \emptyset$. For a hyperplane H that supports P the set $H \cap P$ is called a *face* of P (at times also the sets $\emptyset, P$ are considered faces of P). Faces of polytopes are again polytopes.

For a finite set $X \subseteq \mathbb{R}^n$ with $X = \{x^1, x^2, \ldots, x^m\}$ the *linear hull* is defined by $\mathrm{lin}(X) = \{\lambda^T(x^1, x^2, \ldots, x^m) \colon \lambda \in \mathbb{R}^m\}$. The *affine hull* $\mathrm{aff}(X)$ of X is defined by $\mathrm{aff}(X) = \{\lambda^T(x^1, x^2, \ldots, x^m) \colon \lambda^T\mathbf{1} = 1\}$. The vectors in a finite set $X \subseteq \mathbb{R}^n$ are called *linearly independent* if $\lambda^T(x^1, x^2, \ldots, x^m) = \mathbf{0}$ implies $\lambda = \mathbf{0}$. A finite set of vectors $X = \{x^1, x^2, \ldots, x^m\} \subseteq \mathbb{R}^n$ is called *affinely independent* if $\{x^1 - x^m, x^2 - x^m, \ldots, x^{m-1} - x^m\}$ is linearly independent. The *dimension* of a polytope is just the dimension of its affine hull. A face F of a polytope P is called *facet* if $\dim F = -1 + \dim P$.

Next we come to the more algorithmical problem of *linear programming*. Given $m, d \in \mathbb{N}$, $c \in \mathbb{R}^d, A \in \mathbb{R}^{m \times d}$, a linear program is the task to maximize $c^T x$ with respect to $Ax \leq b$. Whole books have been written about linear (and integer) programming, see Schrijver [SCH89]; Grötschel, Lovász, and Schrijver [GLS93]; and Nemhauser and Wolsey [NW88]. For a long time the *simplex method* was the only algorithm to solve linear programs. But it has been shown by Klee and Minty (1972) that (at least for all known variants of the simplex method) there are examples whose solution require exponential time to solve. A landmark result was Khachian's proof (1979) that the *ellipsoid method* can solve linear programs in polynomial time. This was a theoretically important discovery, but it had little influence on computer programs, as all implementations were (except for pathological examples) slower than the simplex method. Karmarkar's proof (1984) that *interior point methods* can solve linear programs in polynomial time was more influential for practical applications. Nowadays, interior point methods are similarly quick in practice to solve linear programs.

An important feature of the simplex method for many applications is that it always returns an optimal *vertex*. Interior point methods do not guarantee this. But by choosing a random direction in the optimal face and then reoptimizing in it, one obtains with high probability an optimal vertex. Also, by iteratively perturbing the objective function it is possible to obtain an optimal vertex in polynomial time. Finally a method described by Megiddo [MEG91] permits to obtain a primal dual optimal solution in polynomial time.

Integer Programming is the art to solve linear programs that have additional integrality restrictions on (some of) their variables. H. W. Lenstra, Jr. (1983) showed that integer programming in fixed dimension is solvable in polynomial time; a result of Karp [KAR72] however shows, that the task to find a $0-1$-vector that satisfies $Ax = b$ is already $\mathbb{NP}$-complete.

An important special case in which integer programming is just as easy as linear programming occurs if the matrix is totally unimodular. A matrix $A \in \mathbb{R}^{m \times n}$ is called *totally unimodular* (abbreviated by TU) if each subdeterminant of A is in $\{0, \pm 1\}$. In particular, if $A \in \mathbb{R}^{m \times n}$ is TU then

$A \in \{0, \pm 1\}^{m \times n}$. Of all the results that are known about totally unimodular matrices we need only the following three results later in this thesis:

1. If A is TU, so is A^T.
2. If each collection of columns of A can be split into two parts so that the sum of the columns in one part minus the sum of the columns in the other part is a vector with entries only 0, $+1$, and -1 then A is TU.
3. If A is TU, b, b', d, d' are integral and $P(A, b, b', d, d') = \{x \in \mathbb{R}^n : b' \leq Ax \leq b \text{ and } d' \leq x \leq d\}$ is not empty, then $P(A, b, b', d, d')$ is an integral polyhedron.

Even though the second criterion in [SCH89, Thm. 19.3(iv)] is cited from Ghouila-Houri [GH62], our understanding of the main theorem of [GH62] is that it requires slightly less to guarantee TU for a matrix.

2.4. Graph Theory and Graph Algorithms

In this section we give a brief introduction into those areas of graph theory that we need later in this thesis. For a classical book on graphs (and hypergraphs) see Berge [BER70]; for news on algorithmical graph theory see Jungnickel [JUN90].

A *graph* G consists of a finite set V of *vertices* and a set E of 2-element subsets of V (called *edges*). Two vertices $u, v \in V(G)$ are called *adjacent* if $\{u, v\} \in E$. A vertex u is *incident* with an edge $e \in E$ if there exists another vertex $v \in V \setminus \{u\}$ with $e = \{u, v\}$. Frequently we will abbreviate the edge $\{u, v\}$ by uv . The *degree* of $v \in V$ in G is the number of edges with which v is incident.

At times—when considering multigraphs—we will permit also that an edge occurs more than once (this requires E to be considered a multiset) or *loops* that are edges of the form $\{v, v\}$ (this requires to speak no longer of edges as subsets of V). But most of the times we disallow multiple edges and loops; so we do not want to be too formal about the necessary formalities.

A *walk* of length k in a graph G is a sequence of vertices $v_0, v_1, \ldots, v_k$ of G such that $\{v_i, v_{i+1}\} \in E$ for $0 \leq i \leq k-1$. A walk is called a *path* if all the v_i are different. A *circuit* is a walk that fulfills additionally $v_0 = v_k$. Finally a *cycle* is a circuit such that all vertices v_i for $0 \leq i \leq k-1$ are different.

Two more classes of graphs are frequently needed later on. The complete graph K_n on n vertices is a graph with vertex set $V = \{1, 2, \ldots, n\}$ and edge set $E = \{\{i,j\}\colon 0 < i < j \leq n\}$. The cycle C_n has vertex set $V = \{1,2,\ldots,n\}$ and edge set $E = \{\{1,2\},\{2,3\},\ldots,\{n-1,n\},\{n,1\}\}$.

A *digraph* (also known as *directed graph*) D has again a finite set of vertices V. Its set of *arcs* denoted by A (or K) is a subset of $V \times V \setminus \{(i,i)\colon i \in V\}$. Again, one might need to permit multiple arcs or loops (of the form (i,i)). A *directed walk* in a digraph $D = (V,A)$ is a sequence of vertices $v_0, v_1, \ldots, v_k$ of G such that $(v_i, v_{i+1}) \in A$ for $0 \leq i \leq k-1$. The terms *directed path*, *directed circuit*, and *directed cycle* (often also named *dipath*, *dicircuit*, and *dicycle*) are defined accordingly.

A graph or digraph (V,E) is called *bipartite* if there is a partition of V into two sets V_1, V_2 such that there are no arcs (or edges) within V_1 and there are no within V_2.

The *categorical product* $D_1 \cdot D_2$ of two digraphs D_1, D_2 is defined by $V(D_1 \cdot D_2) = V(D_1) \times V(D_2)$ and $A(D_1 \cdot D_2) = \{((u_1,u_2),(v_1,v_2))\colon (u_1,v_1) \in A(D_1) \text{ and } (u_2,v_2) \in A(D_2)\}$, see Babai [Bab95]. Finally we need to define a particular class of digraphs called *lasso* and denoted by L_{l_1,l_2} with $l_1 \geq 0$ and $l_2 > 0$. The vertices of L_{l_1,l_2} are $\{1,2,\ldots,l_1+l_2\}$ and the arcs are $\{(1,2),(2,3),\ldots,(l_1+l_2-1,l_1+l_2),(l_1+l_2,l_1+1)\}$. Notice that $L_{0,k}$ is just a directed k-cycle. The digraph P_l for $l > 0$ will denote the directed path of length l on $l+1$ vertices $0,1,\ldots,l$. The notions of categorical product and lasso will prove to be helpful in Section 5.3 and Subsection 2.4.3.

2.4.1. Single Source Shortest Path Problems. One of the basic algorithmic problems related to graphs is the *single source shortest path problem*. An instance to the shortest path problem is a 3-tuple (G,s,c), where G is a graph (or digraph), s is a vertex of G, and c associates a cost (sometimes also called weight or length) to each edge (or arc) of G. For the purpose of this dissertation we are only concerned with problems with nonnegative weights, so $c : E \longmapsto \mathbb{R}_+$. The task is now to find a path (in the case of a directed graph: to find a dipath) from s to every vertex $t \in V$ so that the sum of the cost of the edges belonging to this path is as small as possible. Unless otherwise specified, we will always assume that for the graph G under consideration the number of vertices is denoted by n and the number of edges by m. Ahuja, Magnanti, and Orlin [AMO93] report that the fastest known shortest path algorithm with a running time of $O(m + n\log n)$ is due to Fredman and Tarjan (1984); this is a Fibonacci heap implementation of Dijkstra's classical algorithm. For sparse graphs—that is, graphs with only very few edges—that fulfill $m = O(n\log n)$ a recent result by Thorup [Tho98] improves the best running time for the *undirected* single source shortest path problem down to $O(m)$. But it seems that his algorithm reaches this linear performance only for *very* large n. Furthermore, we will be concerned with *directed* graphs of *high* density; this gives two more reasons for us not to rely on his result.

2.4.2. All-Pairs Shortest Path Problems. The instances of the *all-pairs shortest path problem* are pairs (G, c) where G is again a (di-)graph and c is the cost function. This time the task is to find shortest paths between *all* pairs of vertices from G. An obvious solution method is to apply the method of Subsection 2.4.1 for every vertex s of G. This yields immediately an algorithm with worst case complexity $O(nm + n^2 \log n)$. This method is preferred for graphs that are not dense, as here the need for only $O(n + m)$ memory is an advantage.

Another method, which is a lot simpler to implement in practice than the Fibonacci heap Dijkstra implementation, is the algorithm of Floyd-Warshall. It has running time $O(n^3)$ so for dense graphs this is competitive with the other method. But its memory requirement is of order n^2, independent of the graph's density.

2.4.3. Shortest Path Problems in Categorical Product Graphs. This subsection is concerned with an important algorithmical building block for separation and recognition methods in Chapter 5. We consider here only digraphs of the form $L_{l_1,l_2} \cdot D$, where D is assumed to be a dense digraph. Let D have vertex set V and arc set A. We always assume that D has a cost function c associated with it. In the new graph $L_{l_1,l_2} \cdot D$ we define the cost of the arc $((i, u), (j, w)) \in A(L_{l_1,l_2} \cdot D)$ (that is, $(i, j) \in A(L_{l_1,l_2})$ and $(u, v) \in A(D)$) to be $c((u, w))$. We are interested in finding all shortest paths from vertices of type $(1, v)$ to vertices of type $(l_1 + 1, v)$. One can solve this by solving an all-pairs shortest path problem as outlined in Subsection 2.4.2. Let $l = l_1 + l_2$. The first method of Subsection 2.4.2 gives a running time of $O(lnm + ln^2 \log(ln))$; the second needs runtime $O(l^3n^3)$. The memory requirements are $O(l(n + m))$ and of order l^2n^2, respectively.

In this subsection we describe a new algorithm with running time $O(n^3(\log l_1 + \log l_2 + \log n))$ and memory requirement $O(n^2)$. We will give the proof by considering shortest path problems for $P_k \cdot D$, $L_{0,k} \cdot D$, and only then for $L_{l_1,l_2} \cdot D$.

First we study the problem to find a shortest path from $(0, u)$ to (k, v) in the digraph $P_k \cdot D$. Let C be the matrix of distances of D, that is

$$(C)_{i,j} = \begin{cases} +\infty & \text{if } (i,j) \notin A, \\ c((i,j)) & \text{if } (i,j) \in A. \end{cases}$$

We can interpret the matrix C as the answer to our particular shortest paths problem for $P_1 \cdot D$ by interpreting the row v of C as labeled by $(0, v)$ and column w of C as labeled by $(1, w)$. Then the entry C_{vw} corresponds to the distance of the shortest path from $(0, v)$ to $(1, w)$ in $P_1 \cdot D$. Next we want to extend this interpretation to $P_k \cdot D$ for $k > 1$.

For two distance matrices C and C' of dimensions $n \times n$ we define the operation $\otimes$ by

$$(C \otimes C')_{ij} = \min_{1 \le k \le n} (C_{ik} + C'_{kj}).$$

Here we use of course $a + \infty = +\infty$ and $\min(a, \infty) = a$ for $a \in \mathbb{R}_+ \cup \{+\infty\}$. It is important to notice that $\otimes$ is associative; therefore there is no trouble in defining the k-th power of C with respect to $\otimes$ for $k > 0$; denote it by $C^{(k)}$. Now it is very easy to verify that the matrices $C^{(k)}$ have again an important interpretation for our particular shortest path problem on $P_k \cdot D$. The entry $(C^{(k)})_{vw}$ is the distance of a shortest path from $(0, v)$ to (k, w) in $P_k \cdot D$. As the binary method of exponentiation (for example given in Knuth [Knu80, Subsection 4.6.3]) requires only that the underlying operation is associative this fact gives immediately an $O(n^3 \log k)$ algorithm to compute $C^{(k)}$. Thereby, our problem to find for every pair of vertices $u, v \in V$ a shortest $(0, u)$–(k, v) path in $P_k \cdot D$ can be solved in time $O(n^3 \log k)$.

Another interpretation of $C^{(k)}$ is that $C^{(k)}_{vw}$ (for $v \neq w$) is the cost of a $(v, 1)$–$(w, 1)$ minimum cost path in $L_{0,k} \cdot D$ of length k (that is a path that meets only $k + 1$ vertices). Now we define the matrix $\tilde{C}$ by

$$(\tilde{C})_{uv} = \begin{cases} C^{(k)}_{uv} & \text{if } u \neq v, \\ 0 & \text{otherwise.} \end{cases}$$

The entry $\tilde{C}_{vw}$ corresponds to the distance of a minimum cost $(v, 1)$–$(w, 1)$ path of length 0 or k.

It is little surprising that we can compute now in $O(n^3 \log n)$ the matrix $\tilde{C}^{(n)}$. For its entry $(\tilde{C}^{(n)})_{vw}$ it is simple to show that it is the length of a minimum cost $(v, 1)$–$(w, 1)$ path in $L_{0,l_2} \cdot D$. So we can compute all pairs minimum cost paths between layer 1 and layer 1 in $L_{0,l_2} \cdot D$ in time $O(n^3(\log l_2 + \log n))$.

Now, we are in good shape to solve the problem we set out to solve: To compute the shortest distance from all vertices of type $(1, v)$ to all vertices of type $(l_1 + 1, v)$ in $L_{l_1,l_2} \cdot D$. Following the same idea as above, it turns out that the matrix $C^{(l_1-1)} \otimes \tilde{C}^{(n)}$ contains in its (u, v)-entry the distance from $(1, u)$ to (l_1+1, v) in $L_{l_1,l_2} \cdot D$. This algorithm takes only time $O(n^3(\log l_1 + \log l_2 + \log n))$, is simple to implement, has for dense graphs a similar complexity as the Fibonacci heap implementation of Dijkstra's algorithm and as the Floyd Warshall algorithm, and requires only $O(n^2)$ memory. But it remains open, which algorithm is the best in practice; of course this depends heavily on the concrete application.

As a corollary to this (not really formulated) theorem we can show that the following optimization problem is solvable in strongly polynomial time:

REMAINDER-RESTRICTED-ALL-PAIRS-SHORTEST-WALK.

Instance:	*A directed graph $D = (V, A)$ with nonnegative weights c and natural numbers q, t, r with $q < t$.*
Output:	*For every pair u, v of vertices the distance of a minimum weight walk between u and v that has length at least r and whose length modulo t is congruent to q.*

So here is our Theorem:

Theorem 2.4.1.
The problem REMAINDER-RESTRICTED-ALL-PAIRS-SHORTEST-WALK *can be solved in time* $O(|V|^3(\log |V| + \log r + \log t))$.

PROOF. Let l_1 be the smallest integer that is at least r and is congruent to q modulo t. Clearly, the encoding length of l_1 is polynomial in the encoding lengths of r, q, t and can be computed from them in polynomial time. Let $l_2 = t$. The main observation, which we need to prove the assertion, is that a minimum weight walk from u to v in D of length at least r and with remainder modulo t congruent to q is in 1–1-correspondence to a shortest $(0, u)$–$((l_1 - 1) + 1, v)$-path in $L_{l_1-1,l_2} \cdot D$. By the previous remarks, we know that we can solve our restricted shortest path problem in $L_{l_1-1,l_2} \cdot D$, with an $O(n^3(\log l_1 + \log l_2 + \log n))$ algorithm. □

2.4.4. Maximum Flow Problems. The *maximum flow problem* models the task to transport as much of a liquid as possible from a given source to a given sink respecting the given capacities on the arcs of the directed graph. Formally speaking an instance is constituted by a 5-tuple (V, A, u, s, t) where V and A are vertex and arc set, respectively; u associates a nonnegative real to every arc; and $s \neq t$ are vertices of (V, A) that model the source and sink of the flow. The u_e are called *capacities.* We associate with every arc (i, j) a variable x_{ij}; this variable describes the amount of flow along this arc. *Flow conservation* requires for all nodes different from s and t that the amount of flow that enters the node also leaves it. The total flow in the network equals the amount of flow that comes out of the node s. Denote this flow by $f = \sum_{j:(s,j)\in A} x_{sj} - \sum_{j:(j,s)\in A} x_{js}$. Next, we state the problem formally.

$$\max \quad f$$

subject to

$$\sum_{j:(i,j)\in A} x_{ij} - \sum_{j:(j,i)\in A} x_{ji} = \begin{cases} f & \text{if } i = s, \\ 0 & \text{if } i \in V \setminus \{s, t\}, \\ -f & \text{if } i = t \end{cases} \quad (\text{for } i \in V)$$

$$0 \leq x_{ij} \leq u_{ij} \text{ for all } (i, j) \in A.$$

For all the applications in this dissertation we can assume that $u_{ij} \in \mathbb{N}_0$. In [AMO93] it is reported that the highest-label preflow-push algorithm solves the maximum flow problem and runs in $O(n^2\sqrt{m})$ as analyzed by Cheriyan and Maheshwari (1989).

For applications to discrete tomography we need a particular type of network. A network (V, A, u, s, t) is called a *unit capacity simple network* if all capacities are 1 and for every vertex $v \in V \setminus \{s, t\}$ holds that either the number of incoming arcs or of departing arcs is at most 1. In [AMO93] is reported that the unit capacity simple network flow problem can be solved in $O(m\sqrt{n})$.

2.4.5. b-Matching and b-Covering Problems. Given an undirected graph $G = (V, E)$ and for every $v \in V$ a bound $b_v \in \mathbb{N}_0$ the MAXIMUM-CARDINALITY-1-CAPACITATED-b-MATCHING optimization problem is the following:

MAXIMUM-CARDINALITY-1-CAPACITATED-b-MATCHING.

Instance:	*A graph $G = (V, E)$ and for every vertex $v \in V$ a capacity $b_v \in \mathbb{N}_0$.*
Output:	*A subset M of E of maximum cardinality such that every vertex $v \in V$ is incident with at most b_v edges of M.*

Recall that we use the shorthand $n = |V|$ and $m = |A|$.

Different variants are known:

1. in the uncapacitated variant M is a multiset, so that M can contain a single edge more than once;
2. in the perfect variant additionally a weight for every edge is given and the task is to find a perfect b-matching (that is one, where every vertex v is incident with exactly b_v edges of M) of maximum weight.

Particularly important is the special case of $b = \mathbf{1}$; the resulting variant of MAXIMUM-CARDINALITY-1-CAPACITATED-b-MATCHING is called MAXIMUM-CARDINALITY-MATCHING. Edmonds was the first who presented an algorithm with polynomial running time of $O(n^4)$ for this problem in 1965. Nowadays (according to [AMO93]) the complexity is improved to $O(m\sqrt{n})$ by Micali and Vazirani (1980). Both algorithms use augmenting paths to improve a matching until it becomes maximal.

As Padberg and Rao [PR82] proved that the separation problem for MAXIMUM-CARDINALITY-1-CAPACITATED-b-MATCHING problem can be solved in polynomial time, it follows with [GLS93] that also MAXIMUM-CARDINALITY-1-CAPACITATED-b-MATCHING is solvable in polynomial time. But this algorithm is only an important tool to prove polynomiality in theory; for all practical purposes it is less helpful as it involves the computationally very difficult—though polynomial—ellipsoid method.

But there is also a purely combinatorial approach to MAXIMUM-CARDINALITY-1-CAPACITATED-b-MATCHING, which we will outline next. This approach is based on a transformation due to Berge [BER70, Chapter 8]. We associate with G of the instance (G, b) a new graph $\bar{G} = (\bar{V}, \bar{E})$ called the *incremental graph*. For $v \in V$ let the set A_v have elements a_v^e for all edges incident with v in G. So $|A_v| = \deg_G(v)$. The set B_v has elements b_v^i for $i = 1, 2, \ldots, \deg_G(v) - b_v$. Assume that all the A_u's and B_v's are disjoint. Set $\bar{V} = \bigcup_{v \in V} (A_v \cup B_v)$. The set of edges is given by

$$\bar{E} = \bigcup_{v \in V} \bigcup_{e = \{u,v\} \in E} \bigcup_{i=1}^{\deg_G(v) - b_v} \{\{a_v^e, b_v^i\}\} \quad \cup \bigcup_{e = \{u,v\} \in E} \{\{a_u^e, a_v^e\}\}.$$

A matching *saturates* a vertex v if v is incident with one edge of M. In [Ber70, Chapter 8, Thm. 1] it is proved, that matchings in $\bar{G}$ that saturate all vertices contained in the B-sets correspond to 1-capacitated-b-matchings of G. The correspondence holds also for matchings and 1-capacitated-b-matchings of *maximum* cardinality.

Now, to obtain a solution algorithm, it is simple to construct directly an initial matching that saturates all the b-vertices in $\bar{G}$. Notice, that applying an augmenting path to a matching that saturates all the b-vertices yields another matching that saturates again all the b-vertices. So the algorithm of Micali and Vazirani can be applied to $\bar{G}$ to obtain in polynomial time a 1-capacitated-b-matching for G. Alternatively, the notion of *alternating paths* can be extended from matchings to 1-capacitated-b-matchings, thereby providing a way to modify the known matching algorithms so that they can directly work on G (without the need to take the detour via $\bar{G}$).

Later, for the approximation algorithms in Chapter 4 we need an unusual covering problem that can be easily solved with the help of a 1-capacitated-b-matching of maximum cardinality.

Minimum-b-Covering.

Instance:	*A graph $G = (V, E)$ and for every vertex $v \in V$ a capacity $b_v \in \mathbb{N}_0$.*
Output:	*A subset M of E of minimum cardinality such that every vertex $v \in V$ is incident with at least b_v edges of M.*

An instance (G, b) of Minimum-b-Covering can be solved by first using Maximum-Cardinality-1-capacitated-b-Matching for (G, b). Denote the solution by M. Then we greedily add unused edges to fulfill the covering condition. But only edges are added, that help to cover at least one of its ends. Denote the set of added edges by A. Clearly, by construction $M \cup A$ is a b-cover for G.

Suppose it is not minimal; then there exists a b-cover C for G with $|C| < |A| + |M|$. We say that an edge $e = \{u, v\}$ of a set of edges F has effect one if the covering condition for F is fulfilled with $>$ at exactly one of the vertices u, v. Now we remove successively edges of effect one from C and put them into a set A'. After finishing this procedure the remaining edges are put into the set M'. Now M' is a b-matching for G, as it contains no more edges of effect one. Because M is a b-matching of maximum cardinality, $|M'| \leq |M|$. So from $|C| < |A| + |M|$ we can conclude that $|A'| < |A|$. But this is impossible, since the edges of M and M' contribute always an effect of two to the b-covering of G, so we obtain from double-counting the identities $2|M| + |A| = \sum_{v \in V} b_v = 2|M'| + |A'|$.

So we can conclude that also MINIMUM-b-COVERING can be solved in polynomial time.

2.4.6. Stable Set Problems. A *stable set* S of a graph (V, E) is a subset of V such that no two vertices of S are adjacent in (V, E). The weighted stable set problem is then to find in the given graph with a weight function on the vertices a stable set of maximum total weight. So the optimization problem STABLE-SET can be formulated in the following way:

STABLE-SET.
Instance: *A graph G and weights w_v for its vertices.*
Output: *A stable set of maximum weight.*

The decision version of STABLE-SET (that is the question: does there exists a stable set of weight at least K for an instance (G, w, K)) is $\mathbb{NP}$-complete as a consequence of the $\mathbb{NP}$-completeness of CLIQUE, which was pointed out in Section 2.2 already. The relation between them (and the reason for their comparable complexity) is that a stable set in G corresponds to a clique in the complement of G, and vice versa. For stable set problems in general, graph-theoretic branch and bound algorithms have been proposed in the literature. For example Babel [BAB93] reports the successful solution of maximum weighted clique problems on random graphs with 500 vertices and 50% edge density; furthermore sparser instances with up to 2000 vertices are solved.

Frequently, the weighted stable set problem (or equivalently the set packing problem) is used only as one relaxation of a partitioning problem. The other relaxation of the partitioning problem is a covering problem. It is very difficult to put additional constraints that are not of packing type into a combinatorial solution algorithm. The polyhedral approach to the stable set problem though, as described for example in Borndörfer's dissertation [BOR97], permits easily to incorporate additional constraints of covering type. For this reason, we want to outline next the polyhedral approach to stable set problems.

In a graph that is not connected, the stable set problem can be solved by solving the problem for the different components separately; hence it suffice to consider stable set problems only for connected graphs.

Let $G = (V, E)$ be a connected, simple graph with $|V| = n \geq 2$ and $|E| = m$. The *incidence vector* of $U \subseteq V$ is $x^U \in \{0,1\}^V$ such that $x_v^U = 1$ if and only if $v \in U$. The *stable set polytope* of G, denoted by $\mathrm{STAB}(G)$, is the convex hull of all incidence vectors of stable sets of G. Some well-known valid inequalities for $\mathrm{STAB}(G)$ include the *trivial inequalities* ($x_v \geq 0$ for $v \in V$), the *cycle inequalities* ($\sum_{v \in C} x_v \leq k$ where C is the vertex-set of a cycle of length $2k+1$), and the *clique inequalities* ($\sum_{v \in K} x_v \leq 1$ where K induces a clique). A clique inequality is called an *edge inequality* if the clique has just two vertices. We define

$$\begin{aligned}
\mathrm{ESTAB}(G) &= \{x \in \mathbb{R}^V : x \text{ fulfills the trivial and edge inequalities}\},\\
\mathrm{CSTAB}(G) &= \{x \in \mathrm{ESTAB}(G)\colon x \text{ fulfills the cycle inequalities}\},\\
\mathrm{QSTAB}(G) &= \{x \in \mathrm{ESTAB}(G)\colon x \text{ fulfills the clique inequalities}\}.
\end{aligned}$$

Let $A_G \in \{0,1\}^{|E| \times |V|}$ be the *edge-vertex-incidence-matrix* of G. We set $(A_G)_{e,i} = 1$ if $i \in e$ and $= 0$ otherwise. Now the stable set problem can be formulated as the following integer program

$$\begin{aligned}
\max \quad & w^T x\\
\text{subject to} \quad &\\
& A_G x \leq \mathbf{1}\\
& x \geq \mathbf{0}\\
& x \in \{0,1\}^{|V|}.
\end{aligned}$$

Computationally, the troublesome part of the preceding problem is the requirement that x is integral, otherwise the machinery of linear programming could solve the problem immediately.

In the polyhedral approach to this problem one tries to find additional strong valid inequalities that describe $\mathrm{STAB}(G)$ more tightly; furthermore it is desirable, that it is possible for given $x^* \in \mathrm{ESTAB}(G)$ (or some other relaxation) to find a violated inequality in this set; this problem is called the *separation problem*. One starts then with solving the LP-relaxation of the problem $\max w^T x$ such that $A_G x \leq \mathbf{1}$ and $x \geq 0$. If the solution is integral we are happy, as we found the optimal solution; otherwise we search for a violated inequality and add it. If no violated inequality can be found, we need to replace the given problem by two problems; assuming that for example x_v^* is fractional, the one problem is obtained by removing v while in the other problem all neighbors of v are removed.

2.5. Five Methods for Two Directions

It has been frequently said in the literature—we have claimed this already too—that the reconstruction problem for two directions is solvable in polynomial time. But most of the time writers are happy if they sketch one method, thereby showing that it is indeed polynomial solvable. Instead, we explain the most common 5 methods to solve the problem for two directions in polynomial time. By putting them next to each other we want to give the reader the opportunity to compare them easily. In particular, we point out for each method whether it can handle problems where the measurements have errors.

For all methods we will use the ground set $G = N_m \times N_n$. Without loss of generality we can assume that $n \leq m$. Suppose a set $F \subseteq G$ is X-rayed along the coordinate directions. Define the resulting row and column sums $r \in \mathbb{N}_0^n, c \in \mathbb{N}_0^m$ by $c_i = |\{j : (i,j) \in F\}|$ for $i \in N_m$ and $r_j = |\{i : (i,j) \in F\}|$ for $j \in N_n$.

We will start with the classical method due to Ryser [RYS63, Chapter 6], [RYS57], also presented by Lorentz [LOR49] and Chang [CHA71].

2.5.1. Ryser's Method. For the method of Ryser [RYS63, Chapter 6] we invest $O(n \log n + m \log m)$ time so that we can assume that the vectors r and c are nonincreasing. Next, the sequence $\bar{c}$ is computed from r by $\bar{c}_i = |\{j : r_j \geq i\}|$ for $i \in N_m$. Ryser proved that if $\sum_{i=1}^k c_i \leq \sum_{i=1}^k \bar{c}_i$ for $k \in N_m$ with equality for $k = m$ then the problem has a reconstruction, that is the row and column sums together are *consistent*.

For doing the reconstruction, we describe now Ryser's method to reconstruct the last column of the problem. After updating the row sums the leftover problem is of size $(m-1) \times n$, and if properly doing this, the row and column sums are still nondecreasingly ordered. Of the m-th column $\{m\} \times N_n$ the positions that correspond to the c_m largest row sums are chosen, if ties occur, the element with the larger row index is chosen. The row sums corresponding to the added atoms are decremented accordingly. The tiebreaking rule guarantees that the updated row sums are again in nondecreasing order. So this method runs in $O(n \log n + m \log m + nm)$. By using a denser output encoding, C. Dürr improved this algorithm so that it has a run time of $O(n \log n + m \log m)$.

It will turn out, that this algorithm is faster than the other presented algorithms. But we are not aware of any variant of this algorithm, that could model more than two directions or that is able to handle erroneous data in any reasonable sense.

2.5.2. Bipartite b-Matching. Another way to solve the reconstruction problem for two directions is to model it as a bipartite b-matching problem. The vertex set is $V = V_1 \cup V_2$ with $V_1 = \{1\} \times N_m$ and $V_2 = \{2\} \times N_n$. The edge set is defined by $E = \{\{(1,i),(2,j)\} \colon i \in N_m \text{ and } j \in N_n\}$. Finally the bounds are

$$b_{k,l} = \begin{cases} c_l & \text{if } k = 1 \text{ and} \\ r_l & \text{if } k = 2. \end{cases}$$

It is simple to see, that for every solution F of the reconstruction problem the set $\{\{(1,i),(2,j)\} \colon (i,j) \in F\}$ is a solution to the maximum cardinality 1-capacitated-b-matching problem on (V,E,b). Similarly, every perfect maximum cardinality 1-capacitated-b-matching M of (V,E,b) provides a solution $\{(i,j) \colon \{(1,i),(2,j)\} \in M\}$ to the reconstruction problem. If the maximum cardinality 1-capacitated-b-matching M of (V,E,b) is not perfect then the reconstruction problem has no solution.

So the methods of Subsection 2.4.5 can be applied to obtain a solution. Reconstruction problems for data with measurement errors can be similarly approached by modifying the graph and then looking for a minimum cost 1-capacitated-b-matching.

2.5.3. Maximum Flow. The idea to solve the reconstruction problem for two directions by a network flow approach has been frequently used in the literature, see Slump and Gerbrands [SG82]; Anstee [Ans83]; and Salzberg, Rivera-Vega, and Rodríguez [SRVR98]. We will briefly summarize it here. The basic idea is to model the reconstruction problem as a maximum flow problem on (V,A,u,s,t). The vertex set is $V = (\{1\} \times N_m) \dot\cup (\{2\} \times N_n) \dot\cup \{s,t\}$. The arc set is defined by $A = \{((1,i),(2,j)) \colon i \in N_m \text{ and } j \in N_n\} \dot\cup \{(s,(1,i)) \colon i \in N_m\} \dot\cup \{((2,j),t) \colon j \in N_n\}$. Finally the capacities are $c_{(1,i),(2,j)} = 1$ for all $i \in N_m$ and $j \in N_n$, $c_{(s,(1,i))} = c_i$ for $i \in N_m$ and $c_{((2,j),t)} = r_j$ for $j \in N_m$. Next, one of the maximum flow algorithms of Subsection 2.4.4 can be used to compute a maximum flow. This can be done in $O((n+m)^2\sqrt{nm})$. If the maximum flow has value less than $\sum_{i=1}^{m} c_i$ or value less than $\sum_{j=1}^{n} r_j$ then the instance (r,c) is inconsistent. Otherwise the set of arcs from vertices of type $(1,i)$ to vertices of type $(2,j)$

corresponds to a solution of the reconstruction problem. It is possible, using standard techniques to transform this network into an unit capacity simple network with $O(nm)$ vertices and $O(n^2m + nm^2)$ arcs. This yields for the specialized algorithm a performance of $O((n^2m + nm^2)\sqrt{nm})$.

Again, it is simple to formulate and solve in this language reconstruction problems for two directions where the measurements have errors.

Salzberg, Rivera-Vega, and Rodríguez [SRVR98] report that they were successful in approximately solving reconstruction problems for more than two directions by using a network model to solve pairs of directions exactly while introducing only small errors into the other directions. It is remarkable though, that they choose for their approach an extension of the bipartite preflow-push algorithm (see [AMO93]) that runs in time $O(\min(n,m)^2nm)$ only. We think that the highest-label preflow-push algorithm would provide immediately superior theoretical performance. Ahuja, Orlin, Clifford, Tarjan [AOST94] report that the theoretical bound for flow algorithms that exploit that the underlying network is bipartite is only then better than the bound for non-bipartite algorithm, if the two vertex sets of the bipartition have different orders of magnitude. But this assumption is by most types of reconstruction problems with fixed direction violated. The assumption can only be fulfilled if the underlying grids grow in different directions with different speeds. But we are not aware of any application where this assumption holds.

2.5.4. Linear Programming. Now we want to argue how to solve the reconstruction problem for two directions with the help of linear programming. For this it is essential to show that the matrix A (see Equation (2.2), Section 2.3), that corresponds to a reconstruction problem with two directions is totally unimodular (TU), see Section 2.3. For the proof we will use that for two directions holds $A = \binom{A_1}{A_2}$. By using the first property of TU of Section 2.3 it suffices to show that $A^T = (A_1^T, A_2^T)$ is TU. Notice that A_1^T contains only a single 1 per row; the same holds for A_2^T. Given a collection J of columns of A^T we partition this collection into two sets J_1, J_2, so that J_1 contains the collection's columns that belong to A_1^T and the other set gets the remaining columns. Adding the columns in J_1 and then subtracting the columns of J_2 we obtain for every entry (corresponding to a row) a value in $\{0, \pm 1\}$ as each row of A^T contains two one, and the corresponding columns cannot be together in J_1 or in J_2. So by the second property of TU we can conclude that A^T is TU. By the third property we know that $P(A, b, b, \mathbf{0}, \mathbf{1})$ is an integral polytope if it is nonempty.

But if the measurements are consistent then there is at least one solution and hence $P(A, b, b, \mathbf{0}, \mathbf{1})$ is nonempty. The vertices of $P(A, b, b, \mathbf{0}, \mathbf{1})$ are then the incidence vectors of solutions to the reconstruction problem. Now the algorithms to solve linear programs and compute a vertex of the optimal region given in Section 2.3 provide a polynomial way to solve the reconstruction problem.

Again it is possible to model and solve the reconstruction problem for data with errors. Even better, it is possible to model reconstruction problems with more directions and to obtain important information about the solutions, see Sections 3.4 and 4.4.

2.5.5. Matroid Intersection. Before we can discuss how to describe the reconstruction problem as a matroid intersection we need to explain what a matroid is. A *matroid* is constituted by a finite ground set V and $\mathcal{M} \subseteq 2^V$, so that

1. $\emptyset \in \mathcal{M}$,
2. if $U \in \mathcal{M}$ and $W \subseteq U$ then $W \in \mathcal{M}$, and
3. if $U, W \in \mathcal{M}$ and $|U| > |W|$ then there exists an element $t \in U \setminus W$ such that $W \cup \{t\} \in \mathcal{M}$.

Sets that are elements of $\mathcal{M}$ are called *independent*. Now the most important observation in connection with discrete tomography is that the reconstruction problem for 1 direction corresponds to a maximization problem over a related matroid. We construct the column matroid $\mathcal{M}_c$ by $V_c = N_m \times N_n$ and $\mathcal{M}_c$ contains exactly those subsets U of V that fulfill $|\{j : (i,j) \in U\}| \leq c_i$ for all $i \in N_m$. The row matroid $\mathcal{M}_r$ has the same ground set but $\mathcal{M}_r$ contains those subsets U of V that fulfill $|\{i : (i,j) \in F\}| \leq r_j$ for all $j \in N_n$. It is very simple to see that $\mathcal{M}_c$ and $\mathcal{M}_r$ fulfill the matroid axioms. Of course for every lattice direction one can build such a matroid. We call matroids of this type *partition matroids*. A subset of V solves a reconstruction problem for that direction, if the subset is independent and there does not exist a larger independent set. Obviously, a set that is independent in $\mathcal{M}_c$ and in $\mathcal{M}_r$ and that is of cardinality $\sum_{i=1}^{m} c_i = \sum_{j=1}^{n} r_j$ (this requires of course that the set is a independent set of maximum cardinality in both matroids) solves the reconstruction problem for two directions. If no such set exist, we can conclude that the given reconstruction problem is inconsistent. The problem to compute a set of maximum cardinality that is independent in two matroids is the cardinality 2-matroid intersection problem.

Usually, a matroid cannot be encoded by listing all its independent sets, as there may be exponentially (in the size of the ground set) many independent sets. But this is unnecessary for our purposes; we will need only a method to decide whether a given set U belongs to the matroid or does not. This can be performed for $\mathcal{M}_c$ by just checking whether all inequalities hold, so it takes $O(mn)$. The same bound holds for $\mathcal{M}_r$. Edmonds [EDM79] and Lawler [LAW75] gave algorithms that can find indeed a set of maximum cardinality that is independent in two matroids. The algorithm uses iterative improvements in the same way as our approximation algorithms in Chapter 4. For 2-matroid intersection it is possible to compute in polynomial time an improvement set (though it could be arbitrary large); but as our problems are $\mathbb{NP}$-hard for 3 and more directions, we have no way to compute improvement sets of arbitrary size in polynomial time. Lawler's algorithm takes $O(nmR^2 + nmR^2nm)$ where R is the number of elements an solution has. This number R is usually unknown, but we know—for obvious reasons—that $R = O(nm)$. So Lawler's algorithm needs for the reconstruction problem with two directions $O(n^3m^3 + n^4m^4)$. This approach, to solve the reconstruction problem by matroid methods, was presented by Gardner, Gritzmann and Prangenberg [GGP96]. On closer inspection it turns out, that Lawler's algorithm is a generalization of the alternating path algorithms for matching.

Next, one wonders of course about the reconstruction problem for three directions. Here we can provide two answers. The first is "yes the problem can be described by an intersection of three partition matroids" but unfortunately the second answer is "no, it would be VERY, VERY surprising if there is an algorithm which can solve the three matroid intersection problem in polynomial time."

Some types of measurement error can be introduced into the two matroid intersection model.

CHAPTER 3

Polytopes and Discrete Tomography

We study polyhedral methods to model and solve problems in discrete tomography. The tomography polytope is studied and some of its facets are described. These results are then implemented into an algorithm that solves problems of size 70×70 on average within 7 minutes. Finally new results on a question, raised by A. Kuba at a 1997 Castle Dagstuhl workshop, are presented.

The results of this chapter are joint work with Peter Gritzmann.

3.1. Definitions and Preliminaries

First we associate with each position p of the set of possible locations $\hat{G}$ a zero-one-variable x_p. Let $\hat{n} = |\hat{G}|$. The measurements are taken in $\hat{m}$ different directions. We refer to subsets of $\hat{G}$ as lines (in later Sections also called *query sets*) to stress the tomographic interpretation, even though we permit these lines to be in fact arbitrary subsets. Each direction $\hat{D}^i \subseteq 2^{\hat{G}}$ is a set of disjoint lines in direction u_i. Here u_i might denote a true direction of $\mathbb{R}^3$ or $\mathbb{R}^2$ or might just serve as another symbol for $\hat{D}^i$.

There are $\hat{d}^i = |\hat{D}^i|$ measurements taken in direction $\hat{D}^i$. Let $\hat{D} = \bigcup_{i=1}^m \hat{D}^i$ be the set of all measured lines. Let $\hat{z}$ be the number of atoms to be placed on $\hat{G}$. The measured data are finally represented by a function $\hat{\phi} : \hat{D} \longmapsto N_{\hat{n}}$. The function $\hat{\phi}$ associates with each line $T \in \hat{D}$ the measured number of atoms $\hat{\phi}(T)$ in a solution $\bar{x}$ along T, namely $\hat{\phi}(T) = \sum_{p \in T} \bar{x}_p$. Henceforth, we will use b_T as a shorthand for $\phi(T)$.

An instance $(\hat{G}, \hat{D}, \hat{b})$ stems from a *geometric* problem if there are direction vectors $u_1, u_2, \ldots, u_m$ in $\mathbb{Z}^3$ or $\mathbb{Z}^2$ which describe the directions $\hat{D}_i$ and if

$$\hat{G} = \bigcup_{T_1 \in \hat{D}^1, \ldots, T_m \in \hat{D}^m} \bigcap_{i=1}^m T_i.$$

We assume for geometric problems that the directions $U = (u_1, u_2, \ldots, u_m)$ are always fixed and that the input is $\hat{b}_T$ only.

Now we are prepared to state the following search problem:

COMBINATORIAL-RECONSTRUCTION.

Instance: *A set of candidate points G, sets of disjoint 'lines' $D^i \subset 2^G$ for $i = 1, \ldots, m$, and the measurements b.*

Output: *An $x \in \{0,1\}^W$ such that $\sum_{p \in T} x_p = b_T$ for all $T \in \bigcup_{i=1}^m D^i$ or the answer* no.

The problem GEOMETRICAL-RECONSTRUCTION(U) is defined analogously, differing only in that the instances (b) describe only the measurements, thereby the grid G and the directions U are not part of the input. (All other geometric problems in this chapter are derived analogously from the corresponding combinatorial problem.) For general problems (i.e. our lines can be arbitrary sets) the COMBINATORIAL-RECONSTRUCTION problem contains the problem of set partitioning, which is $\mathbb{NP}$-hard as Lenstra and Rinnooy Kan [LRK79] have demonstrated. In the more interesting situation for discrete tomography the instances are always geometric. In this case the results of Gardner, Gritzmann, and Prangenberg [GGP99] show that for instances with at least three directions GEOMETRICAL-RECONSTRUCTION(U) is $\mathbb{NP}$-hard; note that U is not part of the input.

It is very natural to describe the reconstruction problem as an integer linear program:

$$\sum_{p \in T} x_p = b_T \text{ for all } T \in D, \tag{3.1a}$$

$$x_p \geq 0 \text{ for all } p \in G, \tag{3.1b}$$

$$x_p \leq 1 \text{ for all } p \in G, \text{ and} \tag{3.1c}$$

$$x \in \{0,1\}^n. \tag{3.1d}$$

Henceforth, we abbreviate the whole equation-system with $Ax = b$ and the subsystems corresponding to D^i with $A^i x = b^i$. Let the convex hull of solutions be $P^=$. Denote with $Q^=$ its LP-relaxation. Notice, that by the results of Subsection 2.5.4 for $m \leq 2$ holds $P^= = Q^=$.

The reconstruction problem is a feasibility version of a generalized set partitioning problem. The classical set partitioning problem is well studied [BP72, BP75, BP76, CY92, BGL+92, HP93, ANS95, Tes94, Bor97] due to its importance for crew-scheduling and other traffic problems.

Fishburn, Schwander, Shepp, and Vanderbei [FSSV97] first studied $Q^=$ in the geometric case. They interpret the fractional solutions as 'characteristic vectors' of fuzzy sets. This concept proves helpful if almost complete LP-invariance (that is, that $Q^=$ has very low dimension) is given. But with no LP-invariance at all, no new information can be gained by this approach. So we need to study the ILP itself.

Notice that the previous description as an ILP might be highly redundant. For example for $T \in \hat{D}^i$ with $\hat{b}_T = 0$ we have $x_p = 0$ for all $p \in T$. Therefore we can delete the points of $\hat{G}$ which are contained in such a line T of zero-measure and set $G_{new} = \hat{G} \setminus T$ and $D^i_{new} = \{S \setminus T \colon S \in \hat{D}^i \text{ and } S \cap T \neq \emptyset\}$, call this a 0-reduction. In the same way lines T of full-measure with $\hat{b}_T = |T|$ imply directly $x_p = 1$ for all p in this T. So we can delete the corresponding points and set $D^i_{new} = \{S \setminus T \colon S \in \hat{D}^i \text{ and } S \cap T \neq \emptyset\}$ and $G_{new} = \hat{G} \setminus T$, while the measurements are updated by $b^{new}_{S \cap T} = \hat{b}_S - |S \cap T|$ for all $S \in D_{new} \setminus \{T\}$. This is called a 1-reduction. These reductions can be applied iteratively until the new problem is irreducible. Let G, n, D, A and b denote the parameters of this *irreducible tomography problem.* The construction of an irreducible problem from an initial problem can be done in polynomial time, therefore in the sequel we will always assume to be confronted with reduced problems.

The next problem to define is the search problem of uniqueness:

Combinatorial-Uniqueness.

Instance:	*A set of candidate points G, sets of lines $D^i \subset 2^G$ for $i = 1, \ldots, m$, the measurements b, and one reconstruction $y \in \{0,1\}^G$.*
Output:	*An $\bar{x} \in \{0,1\}^G \setminus y$ such that $\sum_{p \in T} \bar{x}_p = b_T$ for all $T \in \bigcup D^i$ or the answer* no.

The problem Geometrical-Uniqueness(U) is defined analogously. It follows from [GGP99] that for geometric instances with at least three directions the problem of uniqueness is $\mathbb{NP}$-hard. The $\mathbb{NP}$-hardness in the general (i.e. not necessary geometric) case is an easy consequence of the

$\mathbb{NP}$-completeness of set partitioning. To see this, just augment a given instance of set partitioning problem by adding its ground-set G as another set. Obviously choosing the ground-set gives a solution to the new set partitioning problem. But any other solution (given as proof of nonuniqueness) is a solution for the original instance.

For the ILP the question is whether min $y^T x,\ x \in P^=$ is equal to $y^T y$ (the solution is unique) or if the solution is smaller; in the latter case the reconstruction problem has more than one solution. Notice that the $\mathbb{NP}$-completeness of (the decision version of) uniqueness implies that it is already $\mathbb{NP}$-complete to decide, whether $P^=$ has dimension greater than zero.

Finally we have the problem to decide whether (U_0, U_1) is an invariant set for the given measurement. Here $U_0 \subseteq G$ describes a set of positions, where no atom can be placed, while $U_1 \subseteq G$ is the set of positions, where always atoms have to be placed. So we can describe the decision problem of invariance:

COMBINATORIAL-INVARIANCE.

Instance: *A set of candidate points G, sets of disjoint 'lines' $D^i \subset 2^G$ for $i = 1, \dots, m$, the measurements b, and sets (U_0, U_1).*

Question: *Is it true that $x_i = 0$ and $x_j = 1$ for all $x \in P^=$ and all $i \in U_0$, $j \in U_1$?*

(The problem GEOMETRICAL-INVARIANCE(U) is defined analogously, differing only in that the instances are required to be geometric.) This problem is in co-$\mathbb{NP}$, as it is easy to show that if (U_0, U_1) is not a set of invariance there exists a configuration $x \in P^=$ verifying this. Notice that a given problem instance for reconstruction is uniquely solvable if and only if the pair of sets $(\emptyset, \{p \in G : y_p = 1\})$ is invariant. This implies that COMBINATORIAL-INVARIANCE is co-$\mathbb{NP}$-complete.

For the ILP the invariance problem is to show that $P^=$ is contained in the hyperplane $\sum_{p \in U_1} x_p - \sum_{p \in U_0} x_p = |U_1|$. This implies that to decide whether $P^=$ is contained in a subspace of this type is already co-$\mathbb{NP}$-complete.

As for an easier problem consider the problem of *fractional-invariance (LP-invariance)*, where an LP-invariant pair of sets (U_0, U_1) are sets such that $x_i = 0$ and $x_j = 1$ for all $x \in Q^=$ and all $i \in U_0$, $j \in U_1$.

WEAK-COMBINATORIAL-INVARIANCE.

Instance:	*A set of candidate points* G, *sets of lines* $D^i \subset 2^G$, *for* $i = 1, \dots, m$, *and the measurements* b.
Output:	*Maximal sets* (U_0, U_1) *of LP-invariance.*

(The problem WEAK-GEOMETRICAL-INVARIANCE(U) is defined analogously, differing only in that the instances are required to be geometric.) Both problems are polynomially solvable, see Section 3.4.

The previously given examples demonstrate that polyhedral combinatorics provides a useful language to formulate a variety of different problems in discrete tomography. Furthermore, as almost all stated problems are difficult (either they are $\mathbb{NP}$-complete, $\mathbb{NP}$-hard or they are co-$\mathbb{NP}$-complete), and our problems have rather different variants, there is no hope for a single, simple combinatorial algorithm to tackle all these problems. Therefore to study the polyhedral structure of the involved objects allows a unifying approach to all of them.

3.2. The Tomography Polytope

The polytope $P^=$ is an evil beast, it is already intractable to obtain its dimension. Therefore it is not well suited for polyhedral studies. If we want to prove that a certain inequality induces a facet F of $P^=$ we would need to first compute $d = \dim P^=$ to verify $\dim F = d - 1$.

Therefore we introduce another integer linear programming problem:

$$\max \quad \sum_{p \in G} x_p,$$

subject to

$$\sum_{p \in T} x_p \le b_T \text{ for all } T \in D, \tag{3.2a}$$

$$x_p \ge 0 \text{ for all } p \in G, \tag{3.2b}$$

$$x_p \le 1 \text{ for all } p \in G, \tag{3.2c}$$

$$x \in \{0,1\}^G. \tag{3.2d}$$

Let P be the polytope defined as the convex hull of (3.2a)–(3.2d) and Q its LP-relaxation, obtained by dropping constraint (3.2d). Notice that the reconstruction problem is equivalent to finding a solution to (3.2a)–(3.2d) of value z, the latter being the number of atoms. Now the problem has the form of a generalized set packing problem. Set packing problems have been considered in [PAD73, BP76, CHV75]; Tesch [TES94] and Borndörfer [BOR97] studied polyhedral aspects of set packing to solve Dial-a-Ride problems of different types.

Note that the set $\mathcal{I} = \{x \in \{0,1\}^G : Ax \leq b\}$ is *down-monotone* (that is $x \leq y \in \mathcal{I}$ with x binary implies $x \in \mathcal{I}$). Therefore P is a monotone polytope, cf. Hammer, Johnson and Peled [HJP75]. To study P makes a lot more sense than in the case of other problems like the traveling salesman problem. Because trying to get a solution for this packing problem corresponds to 'place' as many atoms in the lattice as permitted. In the case that the measurements have errors it is not a priorily clear, whether the system with '=' has a solution at all, but the problem with '$\leq$' allows still meaningful approximate solutions; for more on approximate solutions for discrete tomography, see Section 4.6.

A finite set V together with a set $\mathcal{J} \subseteq 2^V$ constitutes an *independence system* if $\mathcal{J}$ contains $\emptyset$ and if for every subset S of V with $S \in \mathcal{J}$ all subsets of S are elements of $\mathcal{J}$. Sets $D \notin \mathcal{J}$ are called *dependent* sets and $I \in \mathcal{J}$ are called *independent.* The minimally dependent sets are called *circuits,* and sets $B \subset V$ are called *bases* if they are maximally independent.

The *rank* of a set $U \subseteq V$, with respect to an independence system $(V, \mathcal{J})$, denoted by $\mathrm{rank}_{(V,\mathcal{J})}(U)$, is the size of a largest base of U. For any independence system $(V, \mathcal{J})$ and any subset U of V, the inequality

$$\sum_{p \in U} x_p \leq \mathrm{rank}(U) \tag{3.3}$$

is a valid inequality of $P(\mathcal{J}) = \mathrm{conv}\{x^S : s \in \mathcal{J}\}$, called *rank inequality;* it is said to be *boolean*, since all coefficients are zero-one. The inequality (3.3) is called *canonical* face of $P(\mathcal{J})$ iff $U = V$. A subset U of V is called *closed* if $\mathrm{rank}(U \cup \{p\}) \geq \mathrm{rank}(U) + 1$ for all $p \in V \setminus U$ and U is called *nonseparable* if $\mathrm{rank}(U) \lneqq \mathrm{rank}(T) + \mathrm{rank}(U \setminus T)$ for all nonempty subsets T of U with $T \neq U$. Obviously, if U is separable or is not closed then the rank inequality (3.3) cannot define a facet.

It is an easy observation that $(G, \mathcal{I})$ is an independence system. Hence we can describe Q as an *independence system polytope*, and obtain the following optimization problem:

$$\max \quad \sum_{p \in G} x_p,$$

subject to

$$\sum_{p\in J} x_p \le b_T \text{ for all } J \in \Omega_{b_T+1}(T) \text{ and all } T \in D, \tag{3.4a}$$

$$x_p \ge 0 \text{ for all } p \in G, \tag{3.4b}$$

$$x_p \le 1 \text{ for all } p \in G, \tag{3.4c}$$

$$x \in \{0,1\}^G \tag{3.4d}$$

with $\Omega_k(M) = \{J \subseteq M : |J| = k\}$. If this new problem has an optimal solution of value z then the reconstruction problem has a solution. For every line $T \in D$ every $J \in \Omega_{b_T+1}(T)$ describes in this (possible exponentially sized) problem a circuit of the independence system $(G, \mathcal{I})$. The set of circuits $\mathcal{C}$ forms another hypergraph $(G, \mathcal{C})$.

The polyhedra associated with the *generalized* set packing problem and with the independence system problem are equal. But the LP-relaxation (3.4a)–(3.4c) is worse (larger) than Q. So for computational studies the system (3.2a)–(3.2c) is more attractive as it is the better, tighter approximation of P (it is *contained* in the other LP-relaxation) and needs only few inequalities. While the independence system formulation (3.4a)–(3.4c) is computationally difficult and weak, it is theoretically well studied for different hypergraphs (cliques, odd holes, odd anti-holes, anti-webs, and generalizations); [NT74, Sek83, EJR87, CL88, CL89, Lau89] show that the corresponding rank inequalities define facets. But unfortunately the hypergraph of the tomography problem belongs to none of these classes.

3.2.1. Critical Graph of a Set-System. Given $\mathcal{S} \subset 2^G$ the *critical graph* $H_{\mathcal{S}}(\mathcal{S})$ is defined to be the graph with vertex set G. Two different vertices $p, p' \in G$ are adjacent in the critical graph if there exists a set $I \ni p$ with $I \in \mathcal{S}$ and $I - p + p' \in \mathcal{S}$. The set-system $\mathcal{S}$ is called *equicardinal,* if all elements of $\mathcal{S}$ have the same cardinality. The next lemma is easy to prove.

Lemma 3.2.1.
If the critical graph of the equicardinal set-system $\mathcal{S}$ is connected and G is not empty, then the incidence vectors of $\mathcal{S}$ span a $(|G|-1)$-dimensional hyperplane

Let the critical graph $H_U(\mathcal{I})$ of an independence system $(G, \mathcal{I})$ with rank function $\operatorname{rank}(\cdot)$ and with respect to a subset U of G be the critical graph of the rank-maximal, independent subsets of U. It is always equicardinal.

Proposition 3.2.2 ([LAU89], Thm. 3.2)**.**
Let U be a closed subset of G and assume that the critical graph $H_U(\mathcal{I})$ of $(G,\mathcal{I})$ is connected. Then the rank inequality (3.3)*, $\sum_{p\in U} x_p \le \operatorname{rank}(U)$, induces a facet of the polytope $P(\mathcal{I})$.*

Notice that for canonical faces the closedness condition is fulfilled trivially.

3.2.2. Facts and Facets. Notice that P depends heavily on the measurements b. Therefore we do not have a single, universal polytope P_n for tomography problems on n positions but a family of polytopes $P_{G,D,b}$ (though usually we will drop G, D, b, having in mind always a special instance). This situation contrasts with the traveling salesman problem (TSP), for which a universal TSP-polytope [GP85] for n cities can be formulated. Therefore, we cannot expect any nontrivial inequality to be facet defining for all tomography problems independent of b.

Nevertheless there are facets known. For example for monotone, fulldimensional polyhedra the following holds (cf. Hammer, Johnson and Peled [HJP75]):

Theorem 3.2.3.
For each $p \in G$, the inequality

$$x_p \ge 0$$

is facet-defining for P.

Of course here and in the sequel we will only consider 0-reduced tomography problems, as in this case $(G,\mathcal{I})$ is *normal* ($\{p\} \in \mathcal{I}$ holds for all $p \in G$ and Q and P are fulldimensional). Inequalities with right hand side zero are called *homogeneous.* It is easy to see (cf. [HJP75]) that all homogeneous inequalities for Q are of the form $x_p \ge 0$ for some $p \in G$.

The next natural question is to characterize the structure of nonhomogeneous facets of P. The following theorem from [HJP75] answers this partially:

Theorem 3.2.4.
If the inequality

$$\pi^T x \le \pi_0 \tag{3.5}$$

defines a facet of P and $\pi_0 \neq 0$ then $\pi_0 > 0$ and $0 \le \pi_p \le \pi_0$ for all $p \in G$.

Now one wonders, if there are nontrivial valid inequalities for $P_{G,D,b}$ of type (3.5), which are valid independently of b. This is answered to the negative:

Theorem 3.2.5.
If the nonhomogeneous inequality (3.5) *defines a facet of the* 0*-reduced problem $P_{G,D,b}$ then this inequality either is not valid for all $P_{G,D,b'}$ where b' varies or* (3.5) *is a positive multiple of $x_p \leq 1$ for a proper $p \in G$.*

(Of course the main reason for this theorem to be true is that almost every zero-one-vector is contained in one of these polyhedra; hence only facets of the cube can be face-defining inequalities of all tomography polytopes for given G and D.)

PROOF. Let $y_p = 1$ iff $\pi_p > 0$ and let b' denote the measurements along D of y. If y fulfills (3.5) then $\sum_p \pi_p \leq \pi_0$. As (3.5) defines a facet of $P_{G,D,b}$ there exists a vector $x \in P_{G,D,b}$ with $\pi^T x = \pi_0$ implying $\sum_p \pi_p \geq \pi_0$. Hence we can conclude $\sum_p \pi_p = \pi_0$. This finally shows that (3.5) is a positive linear combination of the inequalities $x_p \leq 1$ for all $p \in G$. Since it defines a facet it can be only a positive multiple of a single inequality $x_p \leq 1$. $\square$

Since $P_{G,D,b} \subseteq P_{G,D,b'}$ for $b \leq b'$ we obtain the following.

Theorem 3.2.6.
All valid inequalities of $P_{G,D,b'}$ are valid for $P_{G,D,b}$ for $b \leq b'$.

By contrast Theorem 3.2.5 shows that the converse is wrong. For $p \in G$ let D_p denote the set of all lines in D containing p. Another easy theorem follows.

Theorem 3.2.7.
For each $p \in G$, the inequality

$$x_p \leq 1$$

is valid for $P_{G,D,b}$; it defines a facet of $P_{G,D,b}$ if and only if $b_T \geq 2$ for all $T \in D_p$.

PROOF. Suppose that $x_p \leq 1$ defines a facet F. Then for every $q \in G \setminus p$ there exists a vector with $x_q = x_p = 1$ in F (otherwise F is contained in the affine subspaces $x_p = 1$ and $x_q = 0$). This shows that for all $T \in D_{p,q}$, the set of lines containing p and q, holds $b_T \geq 2$.

Conversely, suppose that $b_T \geq 2$ for all $T \in D_p$ holds. Now for every $q \in G \setminus \{p\}$ there is a vector that has 1's at positions p and q only and belongs to the face. Additionally the vector with a single 1 at position p is contained in the face. As these vectors are affine independent they span a facet of P. $\square$

Theorem 3.2.8.
For a geometric problem with m directions and a line $T \in D$ with $|\{S \in D_T \cap D_p \colon b_S = 1\}| \leq |T| - b_T$ for all $p \in G \setminus T$ and $D_T = \{S \in D \colon T \cap S \neq \emptyset\}$, the set of all lines intersecting T, the inequality

$$\sum_{p \in T} x_p \leq b_T$$

defines a facet of $P_{G,D,b}$.

PROOF. The independence system of a reduced problem is normal and there is exactly one line that contains more than one element of T therefore $\operatorname{rank}(T) = b_T$. Furthermore, for each $p \in G \setminus T$ there exists a set U_p of cardinality b_T with $U_p \cup \{p\} \in \mathcal{I}$ since $|T \setminus \{S \cap T \colon S \in D_T \cap D_p \text{ and } b_S = 1\}| \geq b_T$. Therefore $\operatorname{rank}(T \cup \{p\}) \geq \operatorname{rank}(T) + 1$; the set T is closed. Furthermore, the critical graph $H_T(\mathcal{I})$ is connected, since for all p, q there exists a set $U \subset T$ of size $b_T - 1$ (because we assumed a reduced problem $b_T < |T|$) with $U \cup \{p\}, U \cup \{q\} \in \mathcal{I}$ and $\operatorname{rank}(U \cup \{p\}) = \operatorname{rank}(U \cup \{q\}) = b_T$. With Proposition 3.2.2 it follows that the rank inequality $\sum_{p \in T} x_p \leq b_T$ induces a facet. $\square$

The first nontrivial class of facets is given by the 3-hole-inequalities. They are instances of the well-known odd hole inequalities of independence systems.

Theorem 3.2.9.
For a geometric problem and three given lines $T_1, T_2, T_3 \in D$ and with subsets $I_i \subseteq T_i$, intersecting pairwise (in three different points q_1, q_2, q_3, where $q_1 = T_1 \cap T_2$, $q_2 = T_2 \cap T_3$, and $q_3 = T_3 \cap T_1$) with $|I_i| = b_{T_i} + 1$, the inequality $\sum_{p \in I_1 \cup I_2 \cup I_3} x_p \leq b_{T_1} + b_{T_2} + b_{T_3} - 2$ defines a face of P. If furthermore

Condition 1: $|T \cap (I_1 \cup I_2 \cup I_3)| \leq b_T$ for all other lines $T \in D \setminus \{T_1, T_2, T_2\}$ and

Condition 2: for all points $p \in G \setminus (T_1 \cup T_2 \cup T_3)$ there is no line $k \neq T_1, T_2, T_3$ through p which intersects $T_1 \cup T_2 \cup T_3$ in more than $b_k - 1$ points

then the inequality defines a facet.

PROOF. First we prove the validity by adding the inequalities $\sum_{p \in I_i} x_p \leq b_{T_i}$ for $i \in \{1, 2, 3\}$ and $x_p \leq 1$ for all $p \in (I_1 \cup I_2 \cup I_3) \setminus \{q_1, q_2, q_3\}$. The sum is

$$2 \sum_{p \in I_1 \cup I_2 \cup I_3} x_p \leq b_{T_1} + b_{T_2} + b_{T_3} + (b_{T_1} + b_{T_2} + b_{T_3} - 3).$$

This sum can be divided by 2 and the resulting inequalities right hand side of $b_{T_1} + b_{T_2} + b_{T_3} - \frac{3}{2}$ can be rounded down. Hence we obtain the valid inequality $\sum_{p \in I_1 \cup I_2 \cup I_3} x_p \leq b_{T_1} + b_{T_2} + b_{T_3} - 2$.

Regarding the facetness we will first show, that $I_1 \cup I_2 \cup I_3$ is closed in $G \setminus R$ where R are the points of all T_i outside all I_i namely $R = (T_1 \cup T_2 \cup T_3) \setminus (I_1 \cup I_2 \cup I_3)$; then we prove that $H_{I_1 \cup I_2 \cup I_3}(\mathcal{I})$ is connected. Finally we will lift the inequality from $G \setminus R$ to G.

Let $W = I_1 \cup I_2 \cup I_3$. The validity of the inequality shows that $\operatorname{rank}(W) \leq b_{T_1} + b_{T_2} + b_{T_3} - 2$. To prove that $\operatorname{rank}(W) \geq b_{T_1} + b_{T_2} + b_{T_3} - 2$ choose a subset U of W by taking $b_{T_i} - 1$ elements from $I_i \setminus \{q_1, q_2, q_3\}$ (This is possible, because $|I_i \setminus \{q_1, q_2, q_3\}| \geq b_{T_i} - 1$.) for $i \in \{1, 2, 3\}$ and $q_1 = T_1 \cap T_2$. Obviously $U \subset W$ and $|U| = b_{T_1} + b_{T_2} + b_{T_3} - 2$ and U fulfills the constraints of T_1, T_2, T_3. The remaining constraints are fulfilled as a consequence of condition 1. To prove closedness of W in $G \setminus R$ consider an arbitrary point $p \in (G \setminus R) \setminus W$ and observe that the set $U \cup \{p\}$ is independent as a consequence of condition 2. Hence $\operatorname{rank}(U \cup \{p\}) \geq \operatorname{rank}(U) + 1$ and W is closed in $G \setminus R$

In $H_W(\mathcal{I})$ the points q_1, q_2, q_3 are adjacent because $U \cup \{q_i\}$ are rank-maximal in W. Furthermore q_1 is adjacent to all $q \in T_1 \setminus \{q_1, q_2, q_3\}$ because we can construct a new set U' as earlier with the additional property that $q \in U'$, $q_3 \in U'$ and $q_2 \notin U'$. Now U' and $(U' \setminus q) \cup q_1$ are two independent and rank-maximal sets. By interchanging the argument for the other combinations of q_i and T_j we can conclude that $H_{G \setminus R}(\mathcal{I})$ is connected. Now we can invoke Proposition 3.2.2 to conclude that the inequality defines a facet for the problem restricted to $G \setminus R$.

Therefore we know that $|G \setminus R|$ vectors with $|G \setminus R|$ entries exist which give a nonsingular $|G \setminus R| \times |G \setminus R|$ matrix Q. To finish the proof we will construct feasible incidence vectors x with $\sum_{i \in R} x_i = 1$ each of which fulfills $x_i = 1$ for a different $i \in R$ to augment Q to a fullrank $|G| \times |G|$ matrix. Notice that for each $p \in R$ the set $U \cup \{p\}$ is feasible and contained in the face. So we can augment Q to the following matrix

$$Q' = \begin{pmatrix} Q & \mathbf{0} \\ U & \mathbf{I} \end{pmatrix}$$

where $\mathbf{0}$ is a $|G \setminus R| \times |R|$ matrix of zeroes, $\mathbf{I}$ is a $|R| \times |R|$ identity matrix, and $(U, \mathbf{I})$ are the new incidence vectors written rowwise. Obviously Q' has full rank, therefore we have shown that the vectors span a facet. $\square$

The complicated conditions for a 3-hole inequality to define a facet are in the applications usually met as the next (easy to prove) corollary shows.

Corollary 3.2.10.
For a planar geometric problem the condition $b_T \geq 4$ for all $T \in D$ implies that conditions 1 and 2 hold. In this case all 3*-hole inequalities are facet defining.*

These assumptions might appear to be too strong. But in the usual applications many atoms are placed in the picture, so b_T has size of order $b_T/2$ for most l. So $b_T \geq 4$ is not a severe restriction for nonzero-lines. Lines with $b_T = 0$ are cast out by 0-reducing the problem. So we can assume that there are only few constraints with $1 \leq b_T < 3$. Each of these few reduces the dimension of the face only a little. So the faces stay highdimensional.

Theorem 3.2.11.
Separation of the 3*-hole-inequalities is possible in polynomial time.*

PROOF. Suppose a fractional solution x^* is given. Notice that we can consider all different triples of lines T_1, T_2, T_3 intersecting in three different points q_1, q_2, q_3 in polynomial time. Given three lines and a fractional solution x^* a necessary condition for the 3-hole-inequality to be separating is $1 < x^*_{q_1} + x^*_{q_2} + x^*_{q_3} < 2$. This can be verified fast. Choose the sets I_1, I_2, I_3 according to the following rule; let I_i contain q_i, q_{i-1} and the $b_{T_i} - 2$ largest (with respect to the value of x^*_p) positions $p \in T_i$ of x^*. By this mean the 3-hole-inequality with the largest left-hand-side for fixed $T_1, T_2, T_3, q_1, q_2, q_3$ is constructed. $\square$

Notice that the 3-hole-inequalities belong to the class of $\{0, \frac{1}{2}\}$-Chvátal-Gomory cuts [CF96]. They are already $\{0, \frac{1}{2}\}$-cuts of the LU-relaxation (see

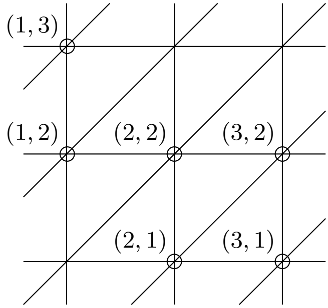


Figure 3.1. Counterexample.

[CF96]) of $Ax \leq b$ and for this larger class of cuts a polynomial separation-algorithm is known. So we gave for some members of the family of $\{0, \frac{1}{2}\}$-cuts of the LU-relaxation sufficient conditions to be facet-defining.

Another question is whether all faces are commonly shared among all 0-reduced polytopes $Q_{G,D,b,z}$ for varying z. Again this question is answered to the negative. For a counterexample (see Figure 3.1) consider the following sets G and D. Let $G = \{(1,2),(1,3),(2,1),(2,2),(3,1),(3,2)\}$ and D is determined by the directions $\{(1,0),(1,1),(0,1)\}$. Compute the measurement b from the configuration with atoms at positions $\{(1,2),(1,3),(2,1),(2,2),(3,1)\}$ and consider $Q_{G,D,b,5}$. For this measurement the problem is 0-reduced and the inequality $x_{13} + x_{12} + x_{22} + x_{32} \leq 3$ defines a face. In contrast for the problem $Q_{G,D,b',5}$ coming from a measurement with atoms at $\{(1,2),(1,3),(2,2),(3,1),(3,2)\}$ the inequality $x_{13}+x_{12}+x_{22}+x_{32} \leq 3$ is invalid. So the inequality $x_{13} + x_{12} + x_{22} + x_{32} \leq 3$ defines a face of $Q_{G,D,b,5}$ but is invalid for $Q_{G,D,b',5}$.

Another big class of facets is constituted by the *chain of triangle inequalities.* An example is drawn in Figure 3.2. A chain T_k of k triangles has ground set $G = \{1, 2, \ldots, 2k + 1\}$ and it has the set of lines

$$D = \{\{1, 2, \ldots, k+1\}\} \cup \bigcup_{i=1}^{k} \{\{i, i+k+1\}\} \cup \bigcup_{i=2}^{k+1} \{\{i, i+k\}\};$$

the measurements along all lines of cardinality 2 are 1 and the measurement along the long line is k.

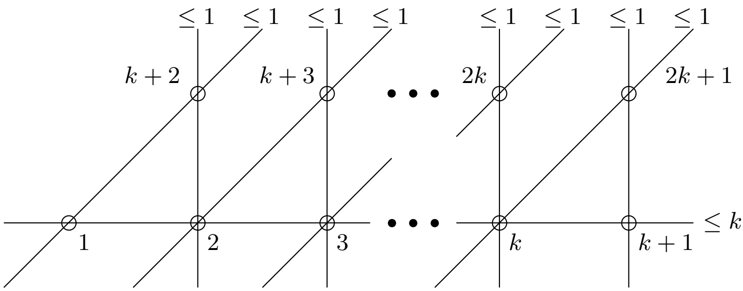


FIGURE 3.2. Example for chain of triangles inequality.

Theorem 3.2.12.
The chain of k triangles T_k has rank k and its rank inequality defines a facet.

PROOF. Notice that $\text{rank}(T_k) = k$ because for each set $U \subseteq T_k$ holds either $|U \cap N_{k+1}| = k$ or $< k$. In the first case, $U \cap (N_{2k+1} \setminus N_{k+1}) = \emptyset$, in the second case $U \cap N_{k+1}$ blocks $|U \cap N_{k+1}|$ of its neighbors in $(N_{2k+1} \setminus N_{k+1})$, hence only $2k+1-(k+1)-|T \cap N_{k+1}|$ of the upper vertices can be chosen.

To prove that it defines a facet notice that all sets with k elements of N_{k+1} are rank-maximal and the sets $U_i = \{k+1+i\} \cup N_{k+1} \setminus \{i, i+1\}$ for $i = 1, \ldots, k$ are rank-maximal too. So $H_{N_{2k+1}}(\mathcal{I})$ is connected. Trivially it is closed, so invoking Proposition 3.2.2 finishes the proof of facetness. $\square$

3.3. The Uniqueness Problem

For the uniqueness problem the following easy to prove theorem provides an ILP formulation.

Theorem 3.3.1.
Consider an optimal and integral solution $\bar{x}$ to $\mathbf{1}^T x$ such that $Ax \leq b$. This solution is unique if the optimum value of $\mathbf{1}^T x$ such that $Ax \leq b$, $\sum_{p \in G} \bar{x}_p x_p \leq \sum_{p \in G} x_p - 1$ and x integral, is strictly smaller than the previous optimum.

For computational results for the uniqueness problem see Chapter 7.

3.4. The Weak Invariance Problem

Theorem 3.4.1.
WEAK-COMBINATORIAL-INVARIANCE *is solvable in polynomial time. Given an interior point solution of* $Q^=$ *all integral components are invariant.*

PROOF. Interior point methods can solve $\max \sum x_p$ s.t. $Ax = b, 0 \leq x \leq 1$ in polynomial time. Fishburn, Schwander, Shepp, and Vanderbei [FSSV97] pointed already out that for an interior point solution x^* the sets $U_0 = \{p \in G\colon x^*_p = 0\}$ and $U_1 = \{p \in G\colon x^*_p = 1\}$ solve LP-invariance. (For a proof notice that an interior-point solution of the problem is in the relative interior of the optimal face ($Q^=$). Now use Caratheodory's theorem.) □

The algorithm of this theorem improves a result of Aharoni, Herman, and Kuba [AHK97]. They gave a criterion to decide for a given solution (but to obtain this solution might be $\mathbb{NP}$-hard already) and a position in it, whether this position is LP-invariant. So to find all LP-invariant positions requires the solution of as many linear programs as there are positions in the instance. By contrast, our method does not need to know a solution and it suffices to solve a single linear program.

3.5. A Solver for Reconstruction

To asses the utility of the polyhedral approach we implemented in C++ a program that solves GEOMETRICAL-RECONSTRUCTION(U). It is based on our class-library[1] for problems of discrete tomography and on CPLEX 6.5 [ILO97].

For three directions $(1, 0), (1, 1), (0, 1)$ and instances of 50% density our program solves quite easily problems of size 70×70 within 7 minutes. For a more complete overview of the running-times see Figure 3.1.

3.6. Greedy Solutions and Greedy Cuts

In the environment of a branch-and-cut-algorithm two ingredients are necessary besides having an LP-problem and an LP-solver:

1. methods to construct a valid inequality from a fractional solution *(separation)* and
2. methods to construct from a given (possibly suboptimal) solution a better one *(augmentation)*.

[1] designed by the author and coded by Jens Zimmermann

Size of Instances	Average Time (in seconds)
50×50	64
60×60	141
70×70	406
80×80	1157
90×90	2551
100×100	4354

Table 3.1. Running-times of the reconstruction algorithm for problems of different sizes (average for 30 instances).

From [GLS93] we know that for a $\mathbb{NP}$-hard optimization problem the separation-problem is $\mathbb{NP}$-hard and it is shown by Schulz, Weissmantel, and Ziegler [SWZ95] that the augmentation-problem is $\mathbb{NP}$-hard. In this light we cannot hope for a polynomial algorithm for separation or augmentation.

Theorem 3.2.11 provides a (very) partial answer, in that it provides an algorithm which (sometimes) finds a violated-constraint.

But still we need something to do about this in practical situations. Usually we can assume that z is larger than $\frac{1}{2}|G| = \frac{1}{2}n$ and that $|D| = O(\sqrt{n})$. This can be exploited to get a good approximate solution.

Theorem 3.6.1.
For a tomography problem with $|D| = O(\sqrt{n})$ it is possible to obtain in polynomial time a feasible solution of value $z - O(\sqrt{n})$.

Proof. With interior point algorithms one can get in polynomial time a primal-dual feasible solution x^*. By choosing a random direction within the optimal face, we can guarantee that x^* is a basic optimal solution x^*. (The same can be done deterministically.) In the feasible solution x^* at least n constraints (but usually a lot more, because Q is strongly degenerate) are active. Obviously at most $|D|$ constraints of $Ax \leq b$ are active, so at least $n - |D|$ constraints of type $x_p \leq 1$ or $x_p \geq 0$ are active. So there are at most $|D|$ fractional positions (that is, $|\{p \in G\colon 0 < x_p < 1\}| \leq |D|$). If we construct $\hat{x}$ as $\hat{x}_p = 1$ iff $x^*_p = 1$ and otherwise $\hat{x}_p = 0$ we round at most $|D|$ fractional variables to zero. So the error is smaller than $|D| = O(\sqrt{n})$. □

Next we propose a method (see Figure 3.3) which solves either the separation-problem or the augmentation-problem (actually more than the augmentation-problem: the optimization-problem) at the same time. Call this method *solve-or-cut.* This method tries at every fractional node of the branch-and-cut-tree first to fix all integral variables and then to solve the residual problem integrally. If it succeeds we found a solution to the original problem; but if the residual optimization fails this implies that it is impossible to complete the fixed part to a fully integral solution. So in the second case we know that at least one of the 1's in the original solution is wrong. This can be expressed as a valid inequality that is violated by the original solution.

procedure *solve-or-cut*

(1) compute a feasible integral solution x from a given fractional solution x^* by use of the rounding-theorem 3.6.1.

(2) compute the measurements for a new subproblem $\mathcal{I}'$, where all integral components of x^* are fixed.

(3) compute an optimal integral solution y of $\mathcal{I}'$ (under $Cy \leq d$).

(4) **if** $\sum_p x_p + \sum_p y_p = z$ **then stop** (found an optimal solution);

(5) **else** Let $\pi_p = 1$ for $x_p^* = 1$, $\pi_p = 0$ for $x_p^* = 0$ and $b_\pi = \sum x_p^* - 1$.
stop ($\pi^T z \leq b_\pi$ is a separating hyperplane).

end *solve-or-cut*

FIGURE 3.3. An algorithm to solve the *solve-or-cut* problem.

At first it does not seem too convincing to try to solve the optimization-problem for $\mathcal{I}$ (perhaps with additional cuts $Cx \leq d$) with a branch-and-cut-algorithm which solves in each node a solve-or-cut problem. But in the practically important case of $z = O(n)$ and $|D| + \#\text{cuts} = O(\sqrt{n})$ (the first is fulfilled for 'dense' problems; the second for geometric problems where only few cuts are added) the subproblems, which have to be solved in step 2) in each node, have only $O(\sqrt{n})$ variables. So for our $\mathbb{NP}$-hard problems the subproblems are very easy.

The next question to address is why the returned cut is valid. Notice that if for x where all components in $W = \{p \in G \colon x_p^* \in \{0,1\}\}$ are fixed there exists no y to extend x to an optimal solution, then every optimal solution x_{opt} differs from x in at least one component of W. Therefore holds $\sum \pi_p z_p \neq b_\pi$. But it is trivial to see that $\pi^T z \leq b_\pi$ always holds. So we can conclude that for all optimal solutions holds $\pi^T z \leq b_\pi - 1$. This implies that $\pi^T z \leq b_\pi - 1$ is a valid cut for $P^=$ but is invalid for P (as it cuts some suboptimal vertices off).

3.7. On the Dimensional Gap between the ILP and LP

In this section we are interested only in geometrical tomography problems. Theoretically, it is of course clear, that even the dimensions of $P^=$ and $Q^=$ cannot always be the same, because otherwise, the uniqueness search problem would be solvable in polynomial time. But nevertheless, one is interested of course in concrete examples that illustrate this point.

The first example of this type was given by Vlach [Vla86, Problem 12] who showed for a certain instance of a 3-dimensional reconstruction problem along the coordinate-axes on a $3 \times 4 \times 6$ cube, that $P^= = \emptyset$ but $|Q^=| = 1$ (that is $\dim P^= = -1 < \dim Q^= = 0$).

At the Workshop “Discrete Tomography: Algorithms and Complexity” in Dagstuhl, January 20th–24th, 1997, A. Kuba posed a similar problem, that can be reduced to the task to construct an example for $\dim P^= = 0 < \dim Q^= = 1$. Initially, we managed to show that there is no such example contained in the $3 \times 3 \times 3$ cube. Next, Gritzmann and Wiegelmann [GW] found an example for Kuba's challenge, by presenting a configuration in the $3 \times 7 \times 7$ box with $\dim P^= = 0 < \dim Q^= = 1$. But it remains still open, whether this example is minimal (in the sense that there exists no example that could be embedded into a smaller box). Given, that this example more than doubles the size of Vlach's, one really wonders why there should not be a smaller example.

So we took up the task to establish new lower bounds for the size of examples for $\dim P^= = 0 < \dim Q^= = 1$. Utilizing the particular symmetries of the problem we managed to check all configurations in a $3 \times 3 \times 4$ cube to show that no example can possibly be contained in this cube; the computation needed 79 hours and 3 minutes of CPU-time on an SGI Origin200 with 4 CPUs at 225MHz. Similarly, we established that the $3 \times 2 \times 6$ cube contains no example; this computation took 362 hours and 3 minutes.

Given the high computational cost (time-wise) we did not pursue larger examples. But it seems feasible, by utilizing the new knowledge about smaller configuration, to sieve out more not interesting instances. By using this iterated sieve, one should be able to improve the lower bound at least in one dimension; so it might be feasible, to decide whether there is any example already of the size of Vlach's example.

CHAPTER 4

Approximating Generalized Cardinality Set Packing and Covering Problems

4.1. Introduction

The present chapter studies various algorithms for solving approximatively *generalized* cardinality set packing and set covering problems. By a generalized packing and covering problem we mean problems, where the required capacities are not only 1's but arbitrary positive integers. The bounds known from Hurkens and Schrijver [HS89] and Halldórsson [HAL96] are generalized to this broader class of problems. It is very surprising that the bounds proved later on match those already known for the more restricted problems. These problems are important building blocks for other problems. For example, a specialized version was used in [GVW] to approximate binary images from discrete X-rays; this application is explained in Section 4.6, where evidence of superior computational performance of our algorithms in practice is given too.

We want to point out to the reader that the notion of *generalized* packing and covering is sometimes used for a different problem, where coefficients of -1 are permitted, see for example [CC95]. But for problems with coefficients of -1 (to our knowledge) nothing is known in general about approximability. Instead, in [CC95] conditions are studied which guarantee integrality of the corresponding polytope (and thereby polynomial solvability of the problem).

This chapter is organized as follows:

Section 4.2 provides the basic notation, states the problems and algorithmic paradigms that are most important in the context of the present chapter, and gives a brief overview of our main results.

Section 4.3 studies various polynomial-time iterative improvement strategies for approximating both problems. We derive performance ratios that show that in this model the optimum can be approximated up to a relative error that depends only on the maximal column degree of the instance. The analysis is based on results of [HS89] and [HAL96] for set packing and set covering heuristics.

Section 4.4 shows that for a certain class of "high rank" instances a polynomial time approximation scheme is available. This result should, however, be regarded as a purely theoretical result. While the class of instances it applies to contains the typical real-world instances of discrete tomography the running time of the algorithm is impractical.

In Section 4.5 the results of Section 4.3 are applied to the problem of finding a stable set of maximum cardinality in graphs that have a common prescribed upper bound for the maximum degree of their vertices.

In Section 4.6 finally we apply the techniques of this chapter to obtain approximate reconstructions for problems of discrete tomography. Additionally, we report on the superior computational performance of these algorithms.

The results of this chapter generalize joint work [GVW] with Peter Gritzmann and Markus Wiegelmann. Furthermore we thank Jens Zimmermann for helping with coding the data-structures and algorithms.

4.2. Preliminaries and Overview

4.2.1. Two Optimization Problems. The set packing problem is the problem to choose from a collection of sets as many disjoint sets as possible. Writing the incidence vectors of the different sets as columns of a matrix A, the problem can be equivalently stated as

$$\begin{aligned} &\max \mathbf{1}^T x \text{ s.t.} \\ &Ax \leq \mathbf{1}, \text{ and } x \in \{0,1\}^N, \end{aligned} \tag{4.1}$$

where $\mathbf{1}$ is the all-ones vector. Similarly the set covering problem requires to choose from a given collection of sets as few as possible which cover all elements in the union of the sets. This problem can be stated as

$$\begin{aligned} &\min \mathbf{1}^T x \text{ s.t.} \\ &Ax \geq \mathbf{1}, \text{ and } x \in \{0,1\}^N. \end{aligned} \tag{4.2}$$

But there is a different (transposed) way to look at these problems, by reading the rows of the matrix A as incidence vectors of *query sets*. So an equivalent rewording of the problem (4.1) is to ask for a maximal collection of items (corresponding to columns) so that from each of the given query sets at most one element is chosen. Accordingly, problem (4.2) is to find a minimal set of elements, so that from each of the query sets at least one element is chosen.

Using this query set formulation, it is simple to describe the generalized set packing and set covering problems central to this chapter. Given a collection $\mathcal{T}$ of (different) subsets of a finite ground set G and a function of capacities $\phi: \mathcal{T} \longmapsto \mathbb{N}_0$ the *generalized set packing problem* is defined as follows.

GENERALIZED-SET-PACKING.

Instance:	*A ground set G, a collection of querysets $\mathcal{T} \subseteq 2^G$, and a function of capacities $\phi: \mathcal{T} \longmapsto \mathbb{N}_0$.*
Output:	*A set $F \subseteq G$ of maximal cardinality such that $\lvert F \cap T\rvert \leq \phi(T)$ for all $T \in \mathcal{T}$.*

Equivalently, if the incidence vectors of different $T \in \mathcal{T}$ are the rows of A and b contains the corresponding ϕ's then GENERALIZED-SET-PACKING can be formulated as the integer linear program

$$\begin{aligned} &\max \mathbf{1}^T x \text{ s.t.} \\ &Ax \leq b, \text{ and } x \in \{0,1\}^G. \end{aligned} \tag{4.3}$$

So the generalized set packing problem is to choose a set $V \subseteq G$ of maximal cardinality such that from each query set $T \in \mathcal{T}$ at most $\phi(T)$ elements are selected. Similarly, the *generalized set covering problem* can be defined.

GENERALIZED-SET-COVERING.

Instance:	*A ground set G, a collection of query sets $\mathcal{T} \subseteq 2^G$, and a function of capacities $\phi: \mathcal{T} \longmapsto \mathbb{N}_0$.*
Output:	*A set $F \subseteq G$ of minimal cardinality such that $\lvert F \cap T\rvert \geq \phi(T)$ for all $T \in \mathcal{T}$.*

Equivalently, if the incidence vectors of different $T \in \mathcal{T}$ are the rows of A and b contains the corresponding ϕ's then GENERALIZED-SET-COVERING can be formulated as the integer linear program

$$\begin{aligned} &\max \mathbf{1}^T x \text{ s.t.} \\ &Ax \geq b, \text{ and } x \in \{0,1\}^G. \end{aligned} \tag{4.4}$$

So the generalized set covering problem is to choose a set $V \subseteq G$ of minimal cardinality such that from each query set $T \in \mathcal{T}$ at least $\phi(T)$ elements are selected.

These two problems are complementary to each other in the following sense. The complement $\bar{F} = G \setminus F$ of a solution $F \subset G$ of an instance of one problem is a solution of the instance with complementary candidate function $\bar{\phi}$ defined by $\bar{\phi}(T) = |G \cap T| - \phi(T)$ of the other problem. This reflects the fact that there are two different ways to describe a solution: either by listing its elements or by listing its non-elements. However, as the direct conversion of an approximation result for Generalized-Set-Packing of the form $|V|/|F| \geq \alpha$ (F is an optimal solution and V is some solution) yields a bound $|\bar{V}|/|\bar{F}| \leq \alpha + (1-\alpha)|G|/|\bar{F}|$ for Generalized-Set-Covering that is dependent on the "density" $|F|/|G|$ of an optimal solution in the ground set, bounds for the relative error of one problem are usually not "identical" to bounds for the other. More importantly, our algorithms for Generalized-Set-Covering are actually insertion methods rather than 'dual' deletion methods. Hence we will consider Generalized-Set-Packing and Generalized-Set-Covering separately in Section 4.3.

The problems Generalized-Set-Packing and Generalized-Set-Covering are already $\mathbb{NP}$-hard (see [GJ79, Problems SP3, SP5]) in the restricted case that b is only an all 1's vector. So in general there is no hope to obtain the optimal solution in polynomial time. Hence for large instances it is desirable to obtain at least provably good (if not optimal) solutions in polynomial time. The bounds presented in this study depend on the column degree of the matrix A, but they do not depend on b. The *column degree* m of a matrix A is defined by

$$m = \max_{j \text{ is a column of } A} \sum_{i \text{ is a row of } A} A_{ij}$$

or equivalently $m = \max_{v \in G} |\{T \in \mathcal{T} : T \ni v\}|$. All approximation guarantees proved in this chapter will depend on the column degree of the underlying incidence structure.

The problems Generalized-Set-Packing and Generalized-Set-Covering are polynomially solvable if $m = 1$; in fact a simple, greedy-type algorithm solves them. For $m = 2$, the problems Generalized-Set-Packing and Generalized-Set-Covering are solvable in polynomial time, if $\mathcal{T}$ permits a partition $\mathcal{T} = \mathcal{T}_1 \dot{\cup} \mathcal{T}_2$ so that the sets in $\mathcal{T}_1$ are disjoint and the sets in $\mathcal{T}_2$ are disjoint; in this case the polynomial time solvability follows from the observation that the underlying matrix A is totally unimodular.

4.2.2. Two Basic Algorithmic Paradigms. In this section we introduce two general algorithmic schemes for solving GENERALIZED-SET-PACKING and GENERALIZED-SET-COVERING that provide the framework for the subsequent approximation algorithms studied in Section 4.3.

Essentially, we distinguish two kinds of approximation algorithms here, iterative improvement strategies and LP-based methods via rounding. The former are built on some greedy method that is refined by improvement and matching techniques, while the latter try to exploit the information gained from solving LP-relaxations of the integer linear programs. The emphasis of the present chapter will be on the theoretical analysis of iterative improvement strategies.

In the simplest classes of local search algorithms for GENERALIZED-SET-PACKING and GENERALIZED-SET-COVERING the neighborhood of a set S is defined as the collection of all supersets of S of cardinality $|S|+1$, or of all subsets of cardinality $|S|-1$, respectively, and the choice is based on some greedy strategy (that may or may not use weights for breaking ties).

In order to increase the performance of such iterative insertion or deletion algorithms, one can apply *r-improvements* for $r \in \mathbb{N}_0$ where an r-element $\langle(r+1)$-element$\rangle$ subset of a current feasible solution $F \subset G$ for the given instance of GENERALIZED-SET-PACKING $\langle$GENERALIZED-SET-COVERING$\rangle$ is deleted and $r+1$ $\langle r \rangle$ elements of $(G \setminus F)$ are inserted while maintaining feasibility. A feasible set $F \subset G$ is called *t-optimal* for the given instance of GENERALIZED-SET-PACKING $\langle$GENERALIZED-SET-COVERING$\rangle$ if no r-improvement is possible for any $r \leq t$. Note that 0-optimality agrees with the common greedy-optimality (no element can be inserted without destroying feasibility for GENERALIZED-SET-PACKING and no element can be removed without destroying feasibility for GENERALIZED-SET-COVERING).

The following paradigm comprises a large class of iterative improvement methods for GENERALIZED-SET-PACKING. A similar paradigm can be formulated for GENERALIZED-SET-COVERING.

Paradigm 4.2.1 (Iterative feasible approximation).

- INPUT: *Capacity function ϕ for the given collection $\mathcal{T}$ of query sets.*

- OUTPUT: *A feasible set $F \subset G$ for the given instance of* GENERALIZED-SET-PACKING.
- COMPUTATION:
  Start with $F = \emptyset$ and successively apply r-improvements for $r \leq t$, for some fixed constant $t \in \mathbb{N}_0$ until no further improvement is possible.

As it is not specified how to select the elements for insertion and deletion, Paradigm 4.2.1 is so general and flexible that it covers a large number of algorithms that incorporate promising refinements. For example, the values of the query set (or residual capacities) could be used to express preferences between elements to be chosen.

Another approach for solving GENERALIZED-SET-PACKING and GENERALIZED-SET-COVERING is based on the linear programming relaxation of (4.3) or (4.4). As a rather general paradigm it can be described as follows.

Paradigm 4.2.2 (LP-based approximation)**.**

- INPUT: *Capacity function ϕ for the given collection $\mathcal{T}$ of query sets.*
- OUTPUT: *A feasible set $F \subset G$ for the given instance of* GENERALIZED-SET-PACKING $\langle$GENERALIZED-SET-COVERING$\rangle$.
- COMPUTATION:
  Compute a solution x_0 of the LP-relaxation of (4.3) $\langle$(4.4)$\rangle$.
  Apply rounding techniques to x_0 to obtain an integer feasible solution.

Section 4.4 will show that for certain "dense" classes of instances of GENERALIZED-SET-PACKING and GENERALIZED-SET-COVERING, LP-based approximation leads to a polynomial-time approximation scheme.

4.2.3. Main Results. For GENERALIZED-SET-PACKING the simplest algorithm within the framework of Paradigm 4.2.1 is the plain greedy algorithm which considers the elements of the ground set in an arbitrary order and successively chooses elements. It is already known by [KH78] that for a greedy-optimal solution V holds

$$\frac{|V|}{|F|} \geq \frac{1}{m}.$$

It is natural to try to improve this algorithm by using 1-improvements, 2-improvements, etc. In this case, Theorem 4.3.1 shows that for a t-optimal solution V,

$$\frac{|V|}{|F|} \geq \frac{2}{m} - \epsilon_m(t),$$

where $\epsilon_m(t)$ is given explicitly and approaches 0 exponentially fast.

Theorem 4.3.4 provides worst case guarantees for GENERALIZED-SET-COVERING. Part (a) shows that a simple greedy-type-insertion-algorithm yields a solution U such that

$$|U|/|F| \leq H(m),$$

where $H(m) = 1 + 1/2 + \cdots + 1/m$ is the m-th harmonic number. If additional matching techniques are applied to obtain a stronger optimality condition ("*matching-optimality*"), then

$$|U|/|F| \leq H(m) - 1/6,$$

Theorem 4.3.4(b).

Theorem 4.3.5 (a) shows that the t-optimality of a solution U guarantees that

$$|U|/|F| \leq m/2 + \epsilon_m(t),$$

where again $\epsilon_m(t)$ is given explicitly and tends to 0 exponentially fast. If, finally, the solution is matching-optimal and (what will be defined later) *effect-3-t-optimal* for $t \geq 5$ then

$$|U|/|F| \leq H(m) - 1/3,$$

Theorem 4.3.5 (c). That is, for $m = 3, 4, 5$ the bounds are $\frac{3}{2}$, $\frac{7}{4}$, and $\frac{39}{20}$.

4.3. Performance Guarantees for Iterative Algorithms

4.3.1. Effects. Let $V \subset G$ and $g \in G \setminus V$. The *effect* $e_V(g)$ of g with respect to V is the number of query sets $S \in \mathcal{T}$ containing g for which the capacity $\phi(S)$ is not yet achieved by V, i.e.,

$$e_V(g) = |\{S \in \mathcal{T} \colon S \ni g \text{ and } |S \cap V| < \phi_i(S)\}|.$$

Clearly, the effect of an element is an integer between 0 and and the column degree m. This notion can easily be extended to subsets of $G \setminus V$. More precisely, let $V' \subset G \setminus V$, then the *effect* $e_V(V')$ of V' with respect to V is defined by

$$e_V(V') = \sum_{T \in \mathcal{T}} e_{V,V'}(T),$$

where

$$e_{V,V'}(T) = \begin{cases} |V' \cap T| & \text{if } |(V \cup V') \cap T| \leq \phi(T); \\ \phi(T) - |V \cap T| & \text{if } |V \cap T| < \phi(T) \text{ and} \\ & \ |(V \cup V') \cap T| \geq \phi(T); \\ 0 & \text{if } |V \cap T| \geq \phi(T). \end{cases}$$

Clearly, $e_V(g) = e_V(\{g\})$; also $e_V(V')$ lends itself to a successive evaluation. In fact, if $V' = \{g_1, \ldots, g_l\}$,

$$e_V(V') = \sum_{i=1}^{l} e_{V \cup \{g_1, \ldots, g_{i-1}\}}(g_i).$$

Furthermore,

$$e = \sum_{i=1}^{m} \sum_{T \in \mathcal{T}_i} \phi_i(T)$$

is called the *total effect* of the given instance. Clearly, if L and U are feasible for the given instance of GENERALIZED-SET-PACKING and GENERALIZED-SET-COVERING, respectively, then $m|L| \leq e \leq m|U|$. In particular, if F is a common solution for the same instance of GENERALIZED-SET-PACKING and GENERALIZED-SET-COVERING, then $e = m|F|$.

4.3.2. Algorithms for Generalized-Set-Packing. The following result gives worst case performance guarantees for a wide class of approximation algorithms for GENERALIZED-SET-PACKING that fit into Paradigm 4.2.1.

Theorem 4.3.1.
Let $t \in \mathbb{N}_0$, let V be t-optimal for a given instance of GENERALIZED-SET-PACKING *and let F be an optimal solution for that instance. Then*

$$\frac{|V|}{|F|} \geq \frac{2}{m} - \epsilon_m(t),$$

where

$$\epsilon_m(t) = \begin{cases} \frac{m-2}{m((m-1)^{s+1}-1)} & \text{if } t = 2s; \\ \frac{2(m-2)}{m(m(m-1)^s-2)} & \text{if } t = 2s-1. \end{cases}$$

Observe that $\epsilon_m(t) \to 0$ as $t \to \infty$. To give an impression of how t enters the bound on the right hand side of Theorem 4.3.1 note that for

$t = 0, \ldots, 5$, the values of $2/m - \epsilon_m(t)$ are $\frac{1}{3}$, $\frac{1}{2}$, $\frac{5}{9}$, $\frac{3}{5}$, $\frac{13}{21}$, $\frac{7}{11}$ when $m = 3$ and $\frac{1}{4}$, $\frac{2}{5}$, $\frac{7}{16}$, $\frac{8}{17}$, $\frac{25}{52}$, $\frac{26}{53}$ when $m = 4$.

For the proof of the case $t > 0$ of Theorem 4.3.1 we need the following combinatorial result of Hurkens and Schrijver [HS89, Theorem 1].

Proposition 4.3.2 (Hurkens and Schrijver).
Let $p, q \in \mathbb{N}$, let V be a set of size q and let $E_1, \ldots, E_p$ be subsets of V. Furthermore, let $m, t \in \mathbb{N}$ with $m \geq 3$ such that the following holds:

(i) *Each element of V is contained in at most m of the sets $E_1, \ldots, E_p$.*
(ii) *For any $r \leq t$, any r of the sets among $E_1, \ldots, E_p$ cover at least r elements of V.*

Then

$$\frac{p}{q} \leq \begin{cases} \frac{m(m-1)^s - m}{2(m-1)^s - m} & \text{if } t = 2s - 1; \\ \frac{m(m-1)^s - 2}{2(m-1)^s - 2} & \text{if } t = 2s. \end{cases}$$

It is convenient to regard V and $\mathcal{E} = \{E_1, \ldots, E_p\}$ as a hypergraph $(V, \mathcal{E})$. Clearly, under the hypothesis of *(i)* and *(ii)* there is some bound on the quotient p/q. The bounds given in Proposition 4.3.2, however, are not that obvious and proved by a quite involved induction. (In addition, [HS89] shows that these bounds are tight.)

Let us point out that in [HS89] Proposition 4.3.2 is used to derive bounds for the approximation error of certain set packing heuristics while in [HAL96] it is utilized for set covering. Our subsequent analysis is based on the ideas of these papers.

PROOF OF THEOREM 4.3.1. For a direct proof of the case $t = 0$, note that the effect of V has to be at least $|F|$ because otherwise the effect of $F \setminus V$ with respect to V would be greater than $(m-1)|F|$. In this case some element of F would have effect m and could hence be added to V without violating the constraints of GENERALIZED-SET-PACKING, in contradiction to the assumption. As the effect of V is exactly $m|V|$, the result follows.

Turning now to the general result note first that it suffices to give a proof under the additional assumption that $V \cap F = \emptyset$. Then, the general case follows via a reduction of the candidate functions for every query set $T \in \mathcal{T}$ by $|V \cap F \cap T|$ with the aid of the inequality

$$\frac{|V|}{|F|} \geq \frac{|V| - |V \cap F|}{|F| - |V \cap F|} \text{ for } |V| < |F|.$$

We define a hypergraph $H = (V, \mathcal{E})$ on the vertex set V with exactly $|F|$ hyperedges (one for each element of F) that satisfies the conditions *(i)* and *(ii)* of Proposition 4.3.2. Let $F = \{f_1, \dots, f_p\}$ and $V = \{v_1, \dots, v_q\}$. The family $\mathcal{E}$ of hyperedges is defined by associating for each $k = 1, \dots, p$ with $f_k \in F$ a set $E_k \subset V$ which encodes the conflicts the insertion of f_k would cause with respect to $\{f_1, \dots, f_{k-1}\}$, F, and V.

For each query set $T \in \mathcal{T}$ define a map $\iota_T : F \cap T \mapsto (F \cup V) \cap T$. Let $F \cap T = \{f_{i_1}, f_{i_2}, \dots, f_{i_a}\}$, and $V \cap T = \{v_{j_1}, v_{j_2}, \dots, v_{j_b}\}$.

If $|F \cap T| \leq |V \cap T|$ we set $\iota_T(f_{i_l}) = v_{j_l}$. If $|F \cap T| > |V \cap T|$ let

$$\iota_T(f_{i_l}) = \begin{cases} v_{j_l} & : \text{for } l \leq |V \cap T|, \text{ and} \\ f_{i_l} & : \text{otherwise.} \end{cases}$$

Now we define the improvement set E_f for a given $f \in F$ by

$$E_f = \{\iota_T(f) \colon T \ni f\} \cap V.$$

We show that the assumptions of Proposition 4.3.2 are satisfied for $t' = t + 1$. To verify *(i)* recall that an element $v \in V$ belongs to E_f if and only if there is a query set T_v with $\iota_{T_v}(f) = v$. This can happen only once for each query set containing v, hence v is contained in at most m different sets E_f.

Next, we show that H has Property *(ii)* of Proposition 4.3.2. Assume on the contrary, that there are sets $E_{k_1}, \dots, E_{k_{r+1}}$ that cover at most r elements of V for some $r \leq t$. By choosing r to be minimal with this property, we can assume that $E_{k_1}, \dots, E_{k_{r+1}}$ cover exactly r elements of V.

Let us consider the set

$$S = \big(V \setminus (E_{k_1} \cup \dots \cup E_{k_{r+1}})\big) \cup \{f_{k_1}, \dots, f_{k_{r+1}}\}.$$

We show that the set S is feasible for the given instance of GENERALIZED-SET-PACKING. Let $T \in \mathcal{T}$. If $|F \cap T| \leq |V \cap T|$ we have

$$|S \cap T| \leq |V \cap T| \leq \phi(T).$$

On the other hand, $|F \cap T| > |V \cap T|$ yields

$$|S \cap T| \leq |F \cap T| \leq \phi(T).$$

This shows that S is indeed feasible for the given instance of GENERALIZED-SET-PACKING.

Because S is obtained from V by deleting the r elements of $E_{k_1} \cup \dots \cup E_{k_{r+1}}$ and inserting the $r+1$ elements $\{f_{k_1}, \dots, f_{k_{r+1}}\}$, S facilitates an r-improvement, a contradiction to the assumption of t-optimality of V.

Summarizing, we have seen that *(i)* and *(ii)* of Proposition 4.3.2 hold for H and $t' = t + 1$, and we obtain

$$\frac{p}{q} = \frac{|F|}{|V|} \leq \begin{cases} \frac{m(m-1)^s - m}{2(m-1)^s - m} & : \ t+1 = 2s-1 \\ \\ \frac{m(m-1)^s - 2}{2(m-1)^s - 2} & : \ t+1 = 2s \end{cases}$$

Hence

$$\frac{|V|}{|F|} = \frac{2}{m} - \left(\frac{2}{m} - \frac{|V|}{|F|}\right) \geq \frac{2}{m} - \epsilon_m(t)$$

which yields the assertion. □

Deterministic polynomial time algorithms that meet the requirements of Theorem 4.3.1 include the greedy algorithm (for $t = 0$) or any other algorithm according to Paradigm 4.2.1.

4.3.3. Greedy Type Insertion for Covering. By changing the stopping rule in Paradigm 4.2.1, an algorithm for solving GENERALIZED-SET-PACKING can be extended to an algorithm for solving GENERALIZED-SET-COVERING. Instead of inserting elements into a set $U \subset G$ only as long as all constraints of GENERALIZED-SET-PACKING are satisfied, such an algorithm inserts elements until the constraints of GENERALIZED-SET-COVERING are satisfied for the first time. As one would never insert an element into the set U that has effect 0, any such heuristic approximates GENERALIZED-SET-COVERING by a factor of at most m. This seems to be the dual result to Theorem 4.3.1 for the case $t = 0$ but it is not because the final set U is not 0-optimal in general. In fact, in this section we will study an insertion strategy for GENERALIZED-SET-COVERING that extends solutions of GENERALIZED-SET-PACKING.

Algorithm 4.3.3 (Greedy strategy for GENERALIZED-SET-COVERING)**.**

- INPUT: *Capacity functions ϕ for the given collection $\mathcal{T}$ of query sets.*
- OUTPUT: *A set $U \subset G$ feasible for the given instance of* GENERALIZED-SET-COVERING*.*
- COMPUTATION:
  1. *Initialize $U = \emptyset$ and $l = m$.*
  2. *Repeat the following step until $l = 0$:*
     - 2.1. *Add elements of effect l to U as long as such elements exist.*
     - 2.2. *Decrease l by* 1.

In the sequel it will often be necessary to regard the elements of U as ordered. This underlying order will always be the element insertion order produced by Algorithm 4.3.3.

The performance guarantees given in the next theorem are derived by a careful analysis of the m iterations of Step 2.1 in Algorithm 4.3.3. Further an additional slight refinement of the algorithm is analyzed. This refinement consists of a combined treatment of elements of effect one and two by means of matching techniques. More precisely, for $l = 1, \ldots, m$ let $U_l \subset U$ be the set of elements constructed for the parameter l in Step 2.1. Then, in the modified version, $U_m, \ldots, U_1$ are first constructed by Step 2 of Algorithm 4.3.3, and subsequently, the following computation is appended as Step 3 in order to decrease $|U_1 \cup U_2|$.

3. *Repeat the following procedure until no further improvements occur:*
   3.1. *Define a graph (V, E) on the vertex set $V = \mathcal{T}$ of all query sets as follows: For a vertex $v \in \mathcal{T}_i$, $1 \le i \le m$, define the degree*
   $$b_v = \max\{0, \phi(v) - |(U_3 \cup \cdots \cup U_m) \cap v|\}.$$
   The edges E are given by means of the set $G' = G \setminus (U_3 \cup \cdots \cup U_m)$ in the following way: For $g \in G'$ let $e_g = \{v \in V : g \in v$ and $b_v > 0\}$. (Note, that $|e_g| \le 2$ because there are no elements of effect at least 3 left in G'.) Now construct a minimum b-edge-cover M for (V, E) and add $U_{1,2} = \{g \in G' : e_g \in M\}$ to U.

Algorithm 4.3.3 can be implemented so as to have a polynomial running time. Using e.g. the method outlined in Subsection 2.4.5 to solve MINIMUM-b-COVERING we see, that Step 3.1 can be carried out in polynomial time. A feasible set U (together with an insertion order) which does not allow any further improvements by means of the procedure in Step 3.1 is called *matching-optimal* (with respect to that order). Note, that the iteration of Step 3 terminates in one step, as after one call upon 3.1 no further improvements are possible. This will be different after another refinement of Algorithm 4.3.3 will be appended as Step 3.2 in Subsection 4.3.4.

Theorem 4.3.4.
Let U be a set of elements constructed by Algorithm 4.3.3 and let F be any feasible solution for GENERALIZED-SET-COVERING.

(a) *Then*

$$\frac{|U|}{|F|} \leq 1 + \frac{1}{2} + \cdots + \frac{1}{m} = H(m) < 1 + \log(m).$$

(b) *If U is matching-optimal, e.g. constructed by Algorithm 4.3.3 extended by Step 3, then*

$$\frac{|U|}{|F|} \leq \frac{5}{6} + \frac{1}{2} + \cdots + \frac{1}{m} = H(m) - \frac{1}{6} < \frac{5}{6} + \log(m).$$

The bounds for $|U|/|F|$ in Theorem 4.3.4 (a) are $\frac{11}{6}, \frac{25}{12}, \frac{137}{60}$ for $m = 3, 4, 5$, respectively. In (b) they are $\frac{5}{3}, \frac{23}{12}, \frac{127}{60}$.

PROOF OF THEOREM 4.3.4. *(a)* Let U_l be again the set of elements inserted in Step 2.1 of Algorithm 4.3.3 for parameter l, i.e. the elements which yield an effect of l upon insertion, and let u_l be the cardinality of U_l. The effect e_l of $U_1 \cup \cdots \cup U_l$ with respect to $U_{l+1}, \ldots, U_m$ is given by $e_l = u_1 + 2u_2 + \cdots + lu_l$. On the other hand, we show that e_l is bounded from above by $l|F|$ for some l.

To this end, let e be the total effect to be attained and suppose to the contrary that $e_l > l|F|$. Consider the set $F' = F \setminus (U_{l+1} \cup \cdots \cup U_m)$. The union of F' and $U_{l+1} \cup \cdots \cup U_m$ contains F, which is feasible for the given instance of GENERALIZED-SET-COVERING, and thus has effect e. Therefore, the effect of F' with respect to $U_{l+1} \cup \cdots \cup U_m$ is exactly e_l and hence, by our assumption, greater than $l|F|$. As $|F'| \leq |F|$ this implies by the pigeon hole principle that there is at least one element $g \in F'$ with effect at least $l+1$. This, however, means that the algorithm would have chosen g rather than some element in $U_1 \cup \cdots \cup U_l$ because all these elements have effect at most l with respect to $U_{l+1} \cup \cdots \cup U_m$, a contradiction. Thus

$$e_l = u_1 + 2u_2 + \cdots + lu_l \leq l|F|, \tag{4.5}$$

for $l = 1, \ldots, m$. Denoting the inequality (4.5) for parameter $l \in \{1, \ldots, m\}$ by I_l we consider the positive linear combination

$$\frac{1}{m} I_m + \sum_{l=1}^{m-1} \frac{1}{l(l+1)} I_l \tag{4.6}$$

of $I_1, \ldots, I_m$. Collecting the terms on the left and on the right of (4.6) we obtain

$$\sum_{i=1}^{m} u_i \leq |F| + \sum_{l=1}^{m-1} \frac{1}{l+1} |F|,$$

which is equivalent to the assertion in *(a)*.

The proof of *(b)* uses the same arguments as that of *(a)* with the difference that appending Step 3.1 to Algorithm 4.3.3 allows to improve inequality I_2 to $u_1 + u_2 \leq |F|$ (instead of $u_1 + 2u_2 \leq 2|F|$).

To prove the new inequality, note that the subset $U_{1,2}$ of G' is determined in Step 3.1 as a minimum b-edge-cover of (V, E). By construction it follows that $U = U_{1,2} \cup U_3 \cup \cdots \cup U_m$ is feasible for the given instance of GENERALIZED-SET-COVERING. Moreover, with $F' = F \setminus (U_3 \cup \cdots \cup U_m)$ the set $\{e_g : g \in F'\}$ is also a b-edge-cover of (V, E). Because $U_{1,2}$ is the disjoint union of (the new sets) U_1 and U_2 and is a minimum b-edge-cover it follows

$$|U_{1,2}| = u_1 + u_2 \leq |F'| \leq |F|. \tag{4.7}$$

With this inequality (instead of inequality I_2) we are led to consider a positive linear combination of type (4.6) with the coefficient $1/2$ of I_1 replaced by $1/3$. This reduces the contribution of I_1 to the coefficient of F on the right hand side by $1/6$. As the other factors remain unchanged, the bound of *(b)* follows. □

4.3.4. Covering via r-Improvements. The aim of this subsection is to analyze an additional refinement of Algorithm 4.3.3 by means of r-improvements. The first step on the way to improved bounds is to study the impact of r-improvements separately (Theorem 4.3.5, *(a)*). Afterwards, the additional gain of r-improvements applied to a matching-optimal configuration is considered by appending to Algorithm 4.3.3 the following Step 3.2 for some (fixed) $t \in \mathbb{N}_0$.

3.2. *Apply all r-improvements for $r \leq t$ to $U_1 \cup U_2 \cup U_3$ that decrease U without destroying its feasibility.*

As in this variant r-improvements are applied only to the set $U_1 \cup U_2 \cup U_3$ the resulting algorithm is faster than the pure exchange-algorithm.

Clearly, because $t \in \mathbb{N}$ is a fixed parameter, Step 3.2 can be performed in polynomial time. A trivial upper bound for the running time is $O(|G|^{2t+2})$. The geometry of discrete tomography, however, allows to reduce this bound for many values of t significantly. The reason is that we do not need to consider all pairs of t- and $(t+1)$-subsets of G but only those which satisfy certain compatibility conditions.

A set $U \subset G$ (together with an insertion order) is called *effect-3-t-optimal* (with respect to this order), if it cannot be decreased by the procedure of Step 3.2 (presented earlier), i.e. by any r-improvement, on the elements of effect one, two, and three.

Theorem 4.3.5.
Let F be a minimum solution for a given instance of GENERALIZED-SET-COVERING *and let $t \in \mathbb{N}_0$.*

(a) *Let U be t-optimal for that instance, then*

$$\frac{|U|}{|F|} \le \frac{m}{2} + \epsilon_m(t)\,, \ \ where\ \epsilon_m(t) = \begin{cases} \frac{m(m-2)}{4(m-1)^{s+1}-2m} & : \ if\, t = 2s; \\ \\ \frac{(m-2)}{2(m-1)^{s}-2} & : \ if\, t = 2s-1. \end{cases}$$

(b) *Let $m = 3$ and $t = 2s+1$, $s \in \mathbb{N}$. Furthermore, assume that U is matching-optimal and t-optimal (that is here: effect-3-t-optimal) then,*

$$\frac{|U|}{|F|} \le \frac{7}{5} + \epsilon'(t)\,, \ \ where\ \epsilon'(t) = \begin{cases} \frac{6}{25\cdot 2^{r+1}-15} & : \ if\, s = 2r-1; \\ \\ \frac{2}{5(5\cdot 2^{r}-1)} & : \ if\, s = 2r. \end{cases}$$

(c) *Let $t \ge 5$ and let U be matching-optimal and effect-3-t-optimal then,*

$$\frac{|U|}{|F|} \le \frac{2}{3} + \frac{1}{2} + \cdots + \frac{1}{m} < \frac{2}{3} + \log(m).$$

The values of $m/2+\epsilon_m(t)$ in Theorem 4.3.5 (a) for $m = 3$ and $t = 0, \ldots, 5$ are $3, 2, \frac{9}{5}, \frac{5}{3}, \frac{21}{13}, \frac{11}{7}$ and for $m = 4$ they are $4, \frac{5}{2}, \frac{16}{7}, \frac{17}{8}, \frac{52}{25}, \frac{53}{26}$. The values of $7/5 + \epsilon'(t)$ for $t = 3, 5, 7, 9, 11$ in (b) are $\frac{11}{7}, \frac{3}{2}, \frac{25}{17}, \frac{13}{9}, \frac{53}{37}$. Note that $\epsilon_m(t), \epsilon'_m(t) \to 0$ for $t \to \infty$ for all $m \ge 3$. The upper bound for $|U|/|F|$ in (c) for $m = 3, 4, 5$ are $\frac{3}{2}, \frac{7}{4}, \frac{39}{20}$.

PROOF OF THEOREM 4.3.5. *(a)* is proved by defining a hypergraph $H = (V, \mathcal{E})$ on the vertex set $V = F$ with edges defined for each $g \in U$ that satisfies *(i)* and *(ii)* of Proposition 4.3.2. As in the proof of Theorem 4.3.1 it suffices to prove the result for $U \cap F = \emptyset$. Again, we define a map $\iota_T : U \cap T \mapsto (U \cup F) \cap T$. This time $\iota_T(u)$ encodes the information which element on T is added to compensate the deletion of u. For each query set $T \in \mathcal{T}$ let $U \cap T = \{u_{i_1}, u_{i_2}, \ldots, u_{i_a}\}$, $F \cap T = \{f_{j_1}, f_{j_2}, \ldots, f_{j_b}\}$.

If $|F \cap T| \ge |U \cap T|$ we set $\iota_T(u_{i_l}) = f_{j_l}$ for $l = 1, 2, \ldots, a$. If $|F \cap T| < |U \cap T|$ let

$$\iota_T(u_{i_l}) = \begin{cases} f_{j_l} & : \text{for } l \le |F \cap T|, \text{ and} \\ u_{i_l} & : \text{otherwise.} \end{cases}$$

Now we define the 'improvement sets' E_u for a given $u \in U$ by

$$E_u = \{\iota_T(u) \colon T \ni u\} \cap F.$$

As in the proof of Theorem 4.3.1 the column degree m gives the bound in *(i)* and the t-optimality implies condition *(ii)* of Proposition 4.3.2 for $t' = t + 1$. Thus Proposition 4.3.2 can be applied, and the bound given in *(a)* follows.

In order to prove *(b)*, let $U = U_1 \cup U_2 \cup U_3$ be a partition of U into subsets of elements of effect 1, 2 and 3, respectively. As each element u of U_1 has effect 1 we can associate with it the query set $T(u)$ it contributes to. For $T \in \mathcal{T}$ let

$$U_T = \{u \in U_1 \cap T \colon T = T(u)\}.$$

Because $|U_T| \le \phi(T) \le |F \cap T|$ for $T \in \mathcal{T}$ we can define an injection $\kappa_T : U_T \mapsto F \cap T$. Now $U_1 = \bigcup_{T \in \mathcal{T}} U_T$, and let $\kappa : U_1 \mapsto F$ be the map induced by the injections κ_T. We show that κ is injective. In fact, if there were $u_1, u_2 \in U_1$ with $\kappa(u_1) = \kappa(u_2)$ then $T(u_1) \neq T(u_2)$, whence $(U \setminus \{u_1, u_2\}) \cup \{\kappa(u_1)\}$ was feasible for the given instance of GENERALIZED-SET-COVERING contradicting the 1-optimality of U. It follows that

$$|U_1| = |F_1|, \tag{4.8}$$

where $F_1 = \kappa(U_1)$.

For the set of remaining elements $F_0 = F \setminus F_1$, we use the fact that there is no r-improvement for U for any $r \le 2s + 1$ in order to show

$$|U_2| + |U_3| \le \left(\frac{3}{2} + \epsilon_3(s-1)\right) |F_0|. \tag{4.9}$$

To this end, let us first define the *reduced* capacity functions

$$\gamma(T) = \min\{\phi(T), |F_0 \cap T|\} \text{ for } T \in \mathcal{T}_i,$$

set $U_{2,3} = U_2 \cup U_3$, and note that $U_{2,3}$ is feasible for the instance $I = \{\gamma_1, \gamma_2, \gamma_3\}$ of GENERALIZED-SET-COVERING. Next, we define a hypergraph $H = (F_0, \mathcal{E})$ with $|U_{2,3}|$ edges, again with the aid of maps ι_T for $T \in \mathcal{T}$. This time $\iota_T : U_{2,3} \cap T \mapsto (U_{2,3} \cup F_0) \cap T$, and $\iota_T(u)$ encodes the information which element on T is added to compensate for the deletion of u in the reduced problem. Let $T \in \mathcal{T}$ and $U_{2,3} \cap T = \{u_{i_1}, u_{i_2}, \dots, u_{i_a}\}$, $F_0 \cap T = \{f_{j_1}, f_{j_2}, \dots, f_{j_b}\}$.

If $|F_0 \cap T| \ge |U_{2,3} \cap T|$ we set $\iota_T(u_{i_l}) = f_{j_l}$ for $l = 1, 2, \dots, a$. If $|F_0 \cap T| < |U_{2,3} \cap T|$ let

$$\iota_T(u_{i_l}) = \begin{cases} f_{j_l} : \text{for } l \le |F_0 \cap T|, \text{ and} \\ u_{i_l} : \text{otherwise.} \end{cases}$$

Now we define the 'improvement sets' E_u for a given $u \in U_{2,3}$ by

$$E_u = \{\iota_T(u) \colon T \ni u\} \cap F_0.$$

To obtain (4.9), we want to apply Proposition 4.3.2 to H. Clearly, condition *(i)* of Proposition 4.3.2 holds with $m = 3$. Next, we show that condition *(ii)* holds with parameter s. Assume on the contrary that there are $l+1$ sets $E_{u_{i_1}}, \ldots, E_{u_{i_{l+1}}}$, with $l+1 \leq s$, that cover only l elements $f_1, \ldots, f_l \in F_0$ and let l be minimal with this property.

Let $\hat{U} = \{u_{i_1}, \ldots, u_{i_{l+1}}\}$, $\hat{F} = \{f_1, \ldots, f_l\}$ and set $S = (U_{2,3} \setminus \hat{U}) \cup \hat{F}$. Of course, S results from $U_{2,3}$ via an l-improvement. While $e_{U_{2,3}\setminus\hat{U}}(\hat{U})$ $\langle e_{U_{2,3}\setminus\hat{U}}(\hat{F})\rangle$ denotes the effect of $\hat{U}$ $\langle\hat{F}\rangle$, with respect to $U_{2,3} \setminus \hat{U}$ and the original data (ϕ), and $\bar{e}$ be the corresponding effect-function for the reduced data (γ). We show that

$$e_{U_{2,3}\setminus\hat{U}}(\hat{U}) \leq e_{U_{2,3}\setminus\hat{U}}(\hat{F}) + l + 3. \tag{4.10}$$

Of course, $e_{U_{2,3}\setminus\hat{U}}(\hat{U}) \leq 3l+3$ and, as S is feasible for I, $\bar{e}_{U_{2,3}\setminus\hat{U}}(\hat{U}) = \bar{e}_{U_{2,3}\setminus\hat{U}}(\hat{F})$. Further, it follows from the minimality of l that $\bar{e}_{U_{2,3}\setminus\hat{U}}(\hat{U}) \geq 2l$. In fact, if $\bar{e}_{U_{2,3}\setminus\hat{U}}(\hat{U}) \leq 2l-1$ then there must exist an $f \in \hat{F}$ of effect 1 with respect to $U_{2,3} \setminus \hat{U}$ and the reduced data, hence

$$(U_{2,3} \setminus (\hat{U} \setminus \{u_f\})) \cup (\hat{F} \setminus \{f\}),$$

where u_f is an element of $\hat{U}$ on the query set T that carries the effect of f, would constitute an $(l-1)$-improvement. This contradiction implies that

$$e_{U_{2,3}\setminus\hat{U}}(\hat{F}) + l + 3 \geq \bar{e}_{U_{2,3}\setminus\hat{U}}(\hat{F}) + l + 3 = \bar{e}_{U_{2,3}\setminus\hat{U}}(\hat{U}) + l + 3 \geq 3(l+1),$$

as claimed.

Next we want to lift the l-improvement for $U_2 \cup U_3$ to an r-improvement for $U_1 \cup U_2 \cup U_3$ with $r \leq 2l$. From (4.10) we know that $e_\emptyset\left(\left((U_1 \cup U_{2,3}) \setminus \hat{U}\right) \cup \hat{F}\right) \geq e - (l+3)$. Hence it suffices to add at most $l+3$ suitable elements $\{g_1, g_2, \ldots, g_{l'}\}$ of F_1 to ensure that

$$\left((U_1 \cup U_{2,3}) \setminus \hat{U}\right) \cup \left(\hat{F} \cup \{g_1, g_2, \ldots g_{l'}\}\right)$$

is feasible. Furthermore, the elements $\kappa^{-1}(g_1), \ldots, \kappa^{-1}(g_l)$ can be deleted from U_1 without destroying feasibility, i.e.,

$$\left((U_1 \cup U_{2,3}) \setminus \left(\hat{U} \cup \{h_1, h_2, \ldots h_{l'}\}\right)\right) \cup \left(\hat{F} \cup \{g_1, g_2, \ldots g_{l'}\}\right)$$

is feasible for the (original) data (ϕ). Let $r = l + l'$, then $r \le 2l + 3 \le 2s + 1 = t$. Hence, the existence of this lifted r-improvement contradicts the t-optimality of U. So, Property *(ii)* holds, Proposition 4.3.2 can be applied, and (4.9) follows.

In order to derive the bound of *(b)*, inequality (4.8), matching-optimality (i.e. inequality (4.7)), and the bound $3|F|$ on the total effect of U are combined to obtain

$$3|U| = |U_1| + \underbrace{(|U_1| + |U_2|)}_{\le |F|} + \underbrace{(|U_1| + 2|U_2| + 3|U_3|)}_{\le 3|F|} \le |F_1| + 4|F|. \tag{4.11}$$

Furthermore, inequality (4.9) implies

$$|U| = |U_1| + (|U_2| + |U_3|) \le |F_1| + \left(\frac{3}{2} + \epsilon_3(s-1)\right)|F_0|. \tag{4.12}$$

Multiplying (4.11) with $\frac{1}{2} + \epsilon_3(s-1)$, adding (4.12) and using $|F_0| + |F_1| = |F|$ then gives

$$\left(\frac{5}{2} + 3\epsilon_3(s-1)\right)|U| \le \left(\frac{7}{2} + 5\epsilon_3(s)\right)|F|,$$

which implies assertion *(b)*.

So we proved for $m \ge 3$, $s \in \mathbb{N}$, $t = 2s + 1$, and U a matching-optimal and effect-3-t-optimal set that

$$\begin{array}{rcl} |U_1| & = & |F_1| \quad \text{and} \\ |U_1| + |U_2| + |U_3| & \le & |F| + \left(\frac{1}{2} + \epsilon_3(s-1)\right)|F_0|. \end{array} \tag{4.13}$$

Finally, we turn to assertion *(c)*. First, we form the positive linear combination

$$\frac{1}{m} I_m + \sum_{l=4}^{m-1} \frac{1}{l(l+1)} I_l.$$

of the inequalities (4.5) derived in the proof of Theorem 4.3.4. Collecting terms for $U_1, \ldots, U_m$ yields

$$\frac{1}{4}|U_1| + \frac{2}{4}|U_2| + \frac{3}{4}|U_3| + |U_4| + \cdots + |U_m| \le \left(1 + \frac{1}{5} + \cdots + \frac{1}{m}\right)|F|.$$

Thus it remains to show that

$$\frac{3}{4}|U_1| + \frac{2}{4}|U_2| + \frac{1}{4}|U_3| \le \frac{3}{4}|F|.$$

Because of $5 \le t = 2s + 1$, we can apply (4.13) for $s = 2$. This yields

$$2|U_1| + |U_2| + |U_3| \le 2|F|.$$

Matching-optimality implies again

$$|U_1| + |U_2| \leq |F|,$$

whence addition of these inequalities gives

$$3|U_1| + 2|U_2| + |U_3| \leq 3|F|.$$

This concludes the proof of Theorem 4.3.5. □

4.4. PTAS for Dense Instances

Let $\gamma > 0$ and $\delta \geq 1$ be two constants. For an instance I of Generalized-Set-Packing ⟨Generalized-Set-Covering⟩ let α_I ⟨β_I⟩ denote the LP-optimum of I i.e., the solution of the LP-relaxation (4.3) ⟨(4.4)⟩. Then I is called (γ, δ)-*dense* if

$$\alpha_I \geq \gamma |M_I|^\delta, \qquad \beta_I \geq \gamma |M_I|^\delta,$$

where M_I is the cardinality of $|\mathcal{T}_I|$. Further, let $\mathcal{I}_{(\gamma,\delta)}$ denote the family of all instances of Generalized-Set-Packing ⟨Generalized-Set-Covering⟩ which are (γ, δ)-dense. Note that it can be checked in polynomial time whether a given instance I belongs to $\mathcal{I}_{(\gamma,\delta)}$.

We have the following approximability result, showing that for dense instances we have a *polynomial time approximation scheme* (PTAS).

Theorem 4.4.1.
Let $1 < \delta$ and $\gamma > 0$. Then there exist PTAS for Generalized-Set-Packing ⟨Generalized-Set-Covering⟩ *when restricted to $\mathcal{I}_{(\gamma,\delta)}$.*

Proof. Let $\epsilon > 0$, and set

$$M_0 := \left\lceil \left(\frac{1}{\epsilon\gamma}\right)^{\frac{1}{\delta-1}} \right\rceil.$$

We show that for all instances I with $M_I \geq M_0$ the relative error of a greedily rounded basic LP-solution is smaller than ϵ. As M_0 is a constant, all smaller instances can be solved exactly in constant time, e.g. by complete enumeration.

So, let $I \in \mathcal{I}_{(\gamma,\delta)}$ with $M_I \geq M_0$, and let F be an optimal solution. Further, let F' be a solution for the given instance of Generalized-Set-Packing ⟨Generalized-Set-Covering⟩ obtained from a basic LP-solution by rounding down ⟨up⟩ the fractional values to 0 ⟨1⟩. Then, of course,

$$|F'| \geq \alpha_I - M_I \qquad \langle |F'| \leq \beta_I + M_I \rangle,$$

whence

$$\frac{|F'|}{|F|} \geq \frac{|F'|}{\alpha_I} \geq 1 - \frac{M_I}{\alpha_I} \qquad \left\langle \frac{|F'|}{|F|} \leq \frac{|F'|}{\beta_I} \leq 1 + \frac{M_I}{\beta_I} \right\rangle.$$

Because I is (γ, δ)-dense and $M_I \geq M_0 = \left\lceil \left(\frac{1}{\epsilon\gamma}\right)^{\frac{1}{\delta-1}} \right\rceil$, we have

$$\frac{M_I}{\alpha_I} \leq \frac{M_I}{\gamma M_I^{\delta}} = \frac{1}{\gamma M_I^{\delta-1}} \leq \frac{1}{\gamma M_0^{\delta-1}} \leq \frac{1}{\gamma \left(\left(\frac{1}{\epsilon\gamma}\right)^{\frac{1}{\delta-1}} \right)^{\delta-1}} = \epsilon$$

$\langle$and similarly $M/\beta_I \leq \epsilon\rangle$. Hence

$$\frac{|F'|}{|F|} \geq 1 - \epsilon \qquad \left\langle \frac{|F'|}{|F|} \leq 1 + \epsilon \right\rangle.$$

□

4.5. Application to Stable Set Problem

For this section we apply the general results of Subsection 4.3.2 to the stable set problem for graphs of degree bounded by Δ defined in the following.

STABLE-SET$_\Delta$.

Instance: *A graph G so that no vertex of it has degree greater than Δ.*

Output: *A set that is stable in G and of maximal cardinality.*

In Papadimitriou and Yannakakis [PY91, Thm. 2(c)] is shown that the (there "INDEPENDENT SET-B" called) Problem STABLE-SET$_\Delta$ is $\mathbb{MAX}$ $\mathbb{SNP}$-complete, but they point out the existence of a simple greedy algorithm approximating within $1/\Delta$. Furthermore, in Alon, Feige, Wigderson, and Zuckerman [AFWZ95, Thm. 3.1] is shown, that there exists an $\epsilon > 0$ such that it is $\mathbb{NP}$-hard to approximate STABLE-SET$_\Delta$ within Δ^ϵ on graphs with maximum degree at most Δ.

By contrast, we give here an argument (derived in [HS89] directly from Proposition 4.3.2) demonstrating that for every $\epsilon > 0$ there is a polynomial time algorithm that approximates STABLE-SET$_\Delta$ with performance $\frac{2}{\Delta} - \epsilon$.

To this end let A denote the edge-vertex incidence matrix and notice that the column-degree of $A(G)$ is at most Δ if G has maximal degree Δ. The stable set problem reduces to a Generalized-Set-Packing with $b = \mathbf{1}$. Now a straight application of Theorem 4.3.1 shows that running an algorithm that guarantees upon termination t-optimality of the solution grants a performance of at least $\frac{2}{\Delta} - \epsilon_\Delta(t)$. So given ϵ we can compute t so that $\epsilon_\Delta(t) < \epsilon$. Now a greedy algorithm followed by doing exchanges until t-optimality is reached will always approximate better than ϵ. So we have proved the next theorem.

Theorem 4.5.1.
The problem Stable-Set$_\Delta$ *can for every* $\epsilon > 0$ *be approximated within* $\frac{2}{\Delta} - \epsilon$ *within polynomial time.*

4.6. Application to Discrete Tomography

In the present section we want to specialize the general approximability results of the Sections 4.3 and 4.4 to discrete tomography and report computational results for this application.

Various approaches have been suggested for solving the general reconstruction problem of discrete tomography, and various theoretical results are available; see e.g. [Gri97] for a survey. In the present section we concentrate on approximative solutions by applying the results of Sections 4.3 and 4.4. Even though most of the resulting combinatorial optimization problems are $\mathbb{NP}$-hard, the application of the results in Section 4.3 implies that some (relatively) simple algorithms yield already very good worst-case bounds. As Subsection 4.6.4 will indicate, these algorithms perform even better in computational practice.

Let us close these remarks with a word of warning. Typically, when one is dealing with optimization problems in practice it is completely satisfactory to produce solutions that are close to optimal. For instance, a tour for a given instance of the traveling salesman problem that is off by only a few percent is for many practical purposes almost as good as an optimal tour. This is due to the fact that the particular optimization is typically just part of a much more complex real world task, and the improvement over existing methods is governed by so many much harder to influence factors that a small error in the optimization step does not really matter by any practical means. This is different in the context of our prime application. The relevant measure for the quality of an approximation to a binary image would of course be the deviation from this image. Hence in order to devise the most appropriate objective function one would have to know the underlying solution of the given inverse problem. However, the whole point is of course to find this unknown solution. Hence one can only consider objective functions with respect to which the approximation is evaluated that are based on the given input data. While a good approximation in this sense is close to a solution in that its X-ray images in the given directions are close to those of the original set, the approximating set itself may be off quite substantially. In fact, the inverse discrete problem is ill-posed and it is precisely this property that causes additional difficulties. In particular, if the input data do not uniquely determine the image even a "perfect" solution that is completely consistent with all given data may be quite different from the unknown real object.

Obviously there is more work to be done to handle the ill-posedness of the problem in practice. Hence, the results of this section should be regarded only as a first (yet reassuring!) step in providing a computational tool that is adequate for the real world applications outlined previously. In particular, our approximation algorithms can be used to provide lower bounds in branch-and-cut approaches, that incorporate strategies to handle the nonuniqueness of solutions and the presence of noise in the data.

4.6.1. Two Optimization Problems. For measuring the quality of approximation methods in discrete tomography, we introduce objective functions so as to formulate the RECONSTRUCTION problem as optimization problems. Two very natural such formulations are the following problems Best-Inner-Fit and Best-Outer-Fit.

BEST-INNER-FIT $(S_1, \ldots, S_m)$.
Instance: *Candidate functions* $\phi_1, \ldots, \phi_m$.
Output: *A set* $F \subset G$ *of maximal cardinality such that* $X_{S_i}F(T) \leq \phi_i(T)$ *for all* $T \in \mathcal{T}_i$ *and* $i = 1, \ldots, m$.

The elements of $\mathcal{T}_i$ correspond correspond to lines parallel to S_i on which there is a nonzero measurement. Then the grid G is implicitly defined as the set of points that belong for each direction to a line with nonzero measurement. An element of $\mathcal{T}_i$ is finally the intersection of the corresponding line with G. BEST-INNER-FIT can be formulated equivalently as the integer linear program

$$\text{(4.14)} \qquad \begin{aligned} &\max \mathbf{1}^T x \text{ s.t.} \\ &Ax \le b, \text{ and } x \in \{0,1\}^G, \end{aligned}$$

where $\mathbf{1}$ is the all-ones vector.

The "outer counterpart" of this inner approximation is defined as follows.

Best-Outer-Fit $(S_1, \ldots, S_m)$.

Instance: *Candidate functions* $\phi_1, \ldots, \phi_m$.

Output: *A set* $F \subset G$ *of minimal cardinality such that* $X_{S_i} F(T) \ge \phi_i(T)$ *for all* $T \in \mathcal{T}_i$ *and* $i = 1, \ldots, m$.

Again, the problem is equivalent to an integer linear program, precisely to

$$\text{(4.15)} \qquad \begin{aligned} &\min \mathbf{1}^T x \text{ s.t.} \\ &Ax \ge b, \text{ and } x \in \{0,1\}^G. \end{aligned}$$

Notice that every instance $(\phi_1, \ldots, \phi_m)$ of Best-Inner-Fit $(S_1, \ldots, S_m)$ $\langle$Best-Outer-Fit $((S_1, \ldots, S_m))\rangle$ can be considered also an instance of Generalized-Set-Packing $\langle$Generalized-Set-Covering$\rangle$ by defining $\phi(T) = \phi_i(T)$ for $T \in \mathcal{T}_i$. Furthermore, the number of directions m of an tomography problem provides at the same time the column degree of the corresponding matrix A.

The two problems Best-Inner-Fit and Best-Outer-Fit are complementary to each other in the same way as Generalized-Set-Packing and Generalized-Set-Covering are as described in Subsection 4.2.1.

Let us remark in passing that one can of course consider other kinds of optimization problems related to Reconstruction$(S_1, \ldots, S_m)$. For instance, rather than measuring the approximability in terms of the points inserted into the candidate grid one may count the number of lines on which an X-ray of a solution coincides with the given value of the corresponding candidate function. An intractability result for this kind of approximation can be found in [GPVW98].

Given that Best-Inner-Fit $\langle$Best-Outer-Fit$\rangle$ are just special cases of the problems Generalized-Set-Packing $\langle$Generalized-Set-Covering$\rangle$ of course the Paradigms 4.2.1 and 4.2.2 can be applied too, so the Theorems 4.3.1, 4.3.4, and 4.3.5 carry directly over to solve Best-Inner-Fit and Best-Outer-Fit. Additionally, as back-projection is a possible solution strategy for continuous tomography, similarly one can use back-projection-like weights to express preferences among different candidate positions in Paradigm 4.2.1, see Algorithm 4.6.1. Additionally, connectivity of the solution (in a sense that is justified by the physical structure of the analyzed material) could be rewarded by introducing adjustable weights. Similarly, information from neighboring layers can be taken into account in a layer-wise reconstruction of a 3-dimensional object. In fact, the positive results of Section 4.3 will apply to the general paradigm.

Clearly there are smarter ways to insert points into the grid than by just greedily putting one in when it fits. A more natural strategy is, for example, to apply a back-projection technique, where each candidate point gets a weight based on the X-ray values of all lines through this point. A typical example is given in Algorithm 4.6.1. In this algorithm, a specific direction S_1 is chosen, which dictates the order in which candidate points are considered for insertion into the set of points L that will eventually form V and the set of holes E (that is disjoint from V). For a fixed line T parallel to S_1, each point g on T gets a weight which depends on the number of points still to be inserted and on the number of candidate points still available on the lines $g + S_i$ for $i \geq 2$, cf. Step 2.1. The corresponding ratio is a value in $[0, 1]$. A value of 0 for a line $g + S_i$ indicates, that the point g cannot be inserted into L and a value of 1 indicates that the point must be inserted into L. Therefore, the product over all $m - 1$ other lines is a natural indicator for comparing the relative importance of the points on line T.

Algorithm 4.6.1 (Weighted greedy strategy)**.**

- INPUT: *Candidate functions* $\phi_1, \ldots, \phi_m$ *for the given directions* $S_1, \ldots, S_m$.
- OUTPUT: *A set* $L \subset G$ *feasible for the given instance of* BEST-INNER-FIT.
- COMPUTATION:
  1. *Initialize* $L = E = \emptyset$ *and choose a specific direction, say* S_1.
  2. *For all* $T \in \mathcal{T}_1$ *do:*
     2.1. *For all* $g \in G \cap T$ *determine*

$$w_g = \prod_{i=2}^{m} \frac{\phi_i(g + S_i) - |(g + S_i) \cap L|}{|(G \setminus (L \cup E)) \cap (g + S_i)|}.$$

     2.2. *Sort* $G \cap T$ *according to decreasing weights* w_g, $g \in G \cap T$ *and add the*

$$\min\{\phi_1(g + S_1), |\{g \in G \cap T \colon w_g > 0\}|\}$$

first elements of $G \cap T$ to L and the remaining ones to E.

It is a well-known result already given by Lorentz, see [LOR49], that this strategy (with a proper ordering of the lines) leads to an exact algorithm for $m = 2$ directions for consistent instances in the plane, cf. [RYS63, Chap. 6]. This suggests, that Algorithm 4.6.1 might be substantially better for arbitrary m than the pure greedy algorithm, an expectation that is confirmed by the experiments stated in Subsection 4.6.4.

Let us point out that the solutions produced by the variant of Algorithm 4.6.1 that is obtained by replacing the weights w_g by

$$w'_g = \prod_{i=1}^{m} \frac{\phi_i(g+S_i) - |(g+S_i) \cap L|}{|(G \setminus (L \cup E)) \cap (g+S_i)|}.$$

coincide with the solutions produced by Algorithm 4.6.1. In fact, while w'_g usually differs from w_g, the order of points on a line in direction S_1 produced by these weights are the same. As for variants, we implemented versions of the algorithm where the $\prod$ in the definition of w'_g is replaced by a $\sum$. But these variants performed weaker than the originally given.

Direct application of Theorem 4.4.1 shows that for dense classes of instances of RECONSTRUCTION$(S_1, \ldots, S_m)$, LP-based approximation leads to a polynomial-time approximation scheme.

4.6.2. Sharpness of the Bounds for Best-Inner-Fit. The following examples show, that the bounds given in Theorem 4.3.1 are tight in the worst case already in the most basic situations.

Example 4.6.2.
Let $m \geq 3$ and let $u_1, \ldots, u_m \in \mathbb{Z}^d$ be m pairwise different lattice directions in $\mathcal{E}^d$. Let $F = \{\nu_1 u_1, \ldots, \nu_m u_m\} \subset \mathbb{Z}^d$ for some scaling factors $\nu_1, \ldots, \nu_m \in \mathbb{Z} \setminus \{0\}$. The X-rays of F in the directions $u_1, \ldots, u_m$ are taken as candidate functions for an instance of BEST-INNER-FIT. *If the factors ν_i are chosen so that $G = F \cup \{0\}$ then $V = \{0\}$ is a greedy-optimal solution for* BEST-INNER-FIT. *Of course $\frac{|V|}{|F|} = \frac{1}{m}$, see Figure 4.1.*

Example 4.6.3.
Let $m = 3$ and let $u_1, u_2, u_3 \in \mathbb{Z}^2$ be the directions $(1,0), (0,1), (1,1)$. The X-rays of $F = \{(0,1), (1,1), (2,2), (3,2)\}$ in the directions u_1, u_2, u_3 are

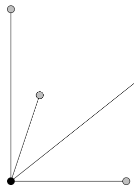


FIGURE 4.1. The greedy bound is tight. (Grey points belong to F, the black point constitutes V.)

taken as candidate functions for an instance of BEST-INNER-FIT. *Then* $V = \{(1,2),(2,1)\}$ *is* 1*-optimal and* $\frac{|V|}{|F|} = \frac{1}{2} = \frac{2}{3} - \epsilon_3(1)$*, see Figure 4.2.*

FIGURE 4.2. The 1-optimality bound is tight for three directions. (Black points denote F in the left picture and V in the right picture)

Example 4.6.4.
Let $m = 4$ *and let* $u_1, u_2, u_3, u_4 \in \mathbb{Z}^2$ *be the directions* $(1,0),(0,1),(1,1),(1,2)$*. The X-rays of* $F = \{(0,0),(1,5),(3,4),(4,3),(5,3)\}$ *in the directions* u_1, u_2, u_3, u_4 *are taken as candidate functions for an instance of* BEST-INNER-FIT*. Then* $V = \{(1,0),(5,5)\}$ *is* 1*-optimal and* $\frac{|V|}{|F|} = \frac{2}{5} = \frac{2}{4} - \epsilon_4(1)$*, see Figure 4.3.*

Example 4.6.5.
Let $m = 5$ *and let* $u_1, \ldots, u_5 \in \mathbb{Z}^2$ *be the directions* $(1,0),(0,1),(1,1),(1,2)$, $(2,1)$ *The X-rays of* $F = \{(0,0),(0,3),(1,3),(2,5),(4,3),(5,4)\}$ *in the directions* $u_1, \ldots, u_5$ *are taken as candidate functions for an instance of*

FIGURE 4.3. The 1-optimality bound is tight for four directions. (Black points denote F in the left picture and V in the right picture)

BEST-INNER-FIT. *Then* $V = \{(2,4),(5,5)\}$ *is* 1*-optimal and* $\frac{|V|}{|F|} = \frac{1}{3} = \frac{2}{5} - \epsilon_5(1)$, *see Figure 4.4.*

FIGURE 4.4. The 1-optimality bound is tight for five directions. (Black points denote F in the left picture and V in the right picture)

4.6.3. Description of the Implementations. We implemented 6 different algorithms for BEST-INNER-FIT. The first algorithm (GreedyA) is the plain greedy algorithm (see Figure 4.5) which considers all positions in a random order and tries to place atoms at these positions. The second algorithm (GreedyB) is a variant of the line following greedy Algorithm 4.6.1 (Figure 4.6). The algorithm chooses a direction with maximal support $|\mathcal{T}_i|$. Suppose—in accordance with the notation in Algorithm 4.6.1—that $i = 1$. The lines $T \in \mathcal{T}_1$ are then considered with respect to decreasing line weights

$$\phi_1(T)/|G \cap T|.$$

procedure *GreedyA*
Calculate a random permutation of all points
For each point in the order of this permutation **do**
    Check whether any line passing through this point is saturated
    If no line is saturated **then**
        Add the point to the solution set
        Update the sums of the lines passing through this point

FIGURE 4.5. The plain greedy solver.

The algorithm usually performs quite well. However, if one considers the 'en block' point insertion procedure successively, i.e. as a point-by-point insertion, then the adapted line weights change and at some point—possibly long before the last point of the block has been inserted—another line might be more profitable. This idea is pursued in a third greedy algorithm (GreedyC) which changes the weights of all lines and uninspected points after a new point is placed; see Figure 4.8. The initial problem with this strategy is of course, that after each insertion a complete search for the next position of maximum weight is necessary. This increases the computation times dramatically. A good data-structure for keeping the points (partially) ordered according to their weights is a heap. After a point insertion, it suffices to update the weights of points on lines through the new point. While a heap can perform this quite efficiently, this procedure is still time consuming because the weights of points may change frequently, without the element even being close to the top of the heap. We decided therefore to use a lazy-update. For this we take the top element of the heap and recompute its weight. Then we compare its stored weight with its actual weight (they might differ due to recent insertions). If the weights are equal, this is still the top element of the heap, and we can try to insert it. If the weights differ, the candidate point gets the new weight and the heap needs to be restructured. After the restructuring we start again with the (new) top element.

The last type of algorithm is the 1-improvement algorithm according to Paradigm 4.2.1. We tried three different variants (ImprovementA, ImprovementB, and ImprovementC) depending on the greedy algorithm (GreedyA, GreedyB, and GreedyC) used first. As the 1-improvement algorithm needs already very long for some instances and the results are very good, we did not dare to implement higher improvement algorithms (like 2-improvement, etc.).

procedure *GreedyB*
Determine a direction with maximal support
Sort the lines parallel to that direction by descending line-weights
For each of these lines (T) in this order **do**
    For each point on T **do**
        Calculate its weight (the product of the line-weights)
    Sort the points on T with respect to descending weights
    For each point in this order **do**
        Check whether any line passing through this point is saturated
        If no line is saturated **then**
            Add the point to the solution set
            Update the sums of the lines passing through this point

FIGURE 4.6. The line following greedy solver.

procedure *GreedyC*
For each point **do**
    Calculate the weight of the point (the product of the relative line capacities)
        and insert it into the heap
While there are still points in the heap **do**
    Find the maximum weight and a corresponding point and remove it from the heap
    Check whether any line passing through this point is saturated
    If no line is saturated **then**
        Add the point to the solution set
        Update the sums of the lines passing through this point

FIGURE 4.7. The dynamically reordering greedy solver.

4.6.4. Performance of the Implemented Algorithms. In this subsection we report on different experiments we conducted with the algorithms described in the previous subsection. We performed several tests for problems of size 20×20 to 500×500, with 2 to 5 directions and of density between $1\%, 5\%, 20\%$, and 50%. Surprisingly, the outcomes are almost independent (up to artifacts) of the density of the instances.

Even though our program can solve problems in three dimensions and on arbitrary crystal-lattices, we decided to present here only results for 2-dimensional problems on the square lattice, as in the physical application all directions belong to a single plane (therefore the problem can be solved in a slice by slice manner); furthermore this restriction should facilitate the comparison with other, less general codes currently under development by various research groups.

Whenever we report either running-times or performances, we report the average of 100 randomly generated instances. We decided here for random

```
procedure Improvement[ABC]
Calculate a solution U according to GreedyA, GreedyB, or GreedyC
Repeat
    For each point (p1) of the candidate grid do
        If p1 ∈ U then continue with the next point
        If no line passing through p1 is saturated then
            Add p1 to U
            Update the sums of the lines passing through p1
            Continue with the next point
        If more than one line passing through p1 is saturated then
            continue with the next point
        For each point (p2) of U on the saturated line do
            For each non-saturated line (T1) through p1 do
                Calculate the line (T2) parallel to T1 passing through p2
                For each point (p3) on T2 not in U do
                    If the lines passing through p3 and
                        not containing p1 or p2 are non-saturated and
                        the line passing through p3 and p1 (if existent)
                        has at least one point not in U then
                        Perform the improvement:
                            Remove p2 from U
                            Add p1 to U
                            Add p3 to U
                        Update the sums of all lines passing through p1, p2 or p3
                        Continue with the next point
Until no improvement was done in the last loop
```

FIGURE 4.8. The improvement solvers.

instances for two reasons. The first reason is that we still lack sufficient experimental data from the physicists. On the other hand, it is typically easy to detect and then eliminate invariant points i.e., points that either must belong to every solution or do not belong to any solution. Because the invariant points carry much of the physical a priori knowledge the reduced problem tends to be quite unstructured.

To obtain a random configuration of prescribed density, we generate a random permutation of the positions of the candidate grid and then place atoms in this order until the described density is reached. After calculating the lines and their sums we discard the configuration itself. Then we preprocessed the problem by calculating the incidence tables, which are necessary for all algorithms. The running-times we report were obtained on an SGI Origin 200 computer with four MIPS R10000 processor at 225MHz with 1GB of main memory by running at most three test programs at the same time.

Note that all instances are consistent. This has two reasons. First, for inconsistent problems we need the exact solution to evaluate the performance of the heuristics. But for the relevant dimensions there are at present no algorithms available that produce exact solutions in reasonable time. The second reason is that the true nature of the error-distribution for the real physical objects has not been experimentally determined by the physicists yet. So it is not clear how to perturb an exact instance to obtain inconsistent problems in a physically reasonable manner.

Now we discuss the results for 50% dense instances. The performance plotted in Figure 4.21 is the quotient of the cardinality of the approximate solution to that of an optimal solution. The closer it is to 1 the better the result is. It turns out, that the larger the problems, the better every algorithm performs in terms of relative errors (see Figure 4.21). Obviously, postprocessing the output of some greedy algorithm with an improvement algorithm cannot decrease the performance (usually it improves the performance). However, it turns out that GreedyB outperforms ImprovementA (for 4 and 5 directions) and that GreedyC performs better than ImprovementB (for 5 directions; for 4 directions they are similar and for 3 directions ImprovementB is better).

The running-times for the algorithms GreedyA and GreedyB are less than 4 seconds for all instances (of size up to 500×500) of density 50%. The application of the 1-improvements to their results increases the running-time to up to 100 seconds. Generally GreedyA and ImprovementA take about half the time of GreedyB and ImprovementB, respectively.

The running-times of GreedyC and ImprovementC increase much faster than those for the other algorithms. Still, they take only up to 1300 seconds. This is long, but in fact, these algorithms provide very close approximations while presently available exact algorithms seem incapable of solving 500×500 problems in less than a century. Furthermore, knowing a solution for a neighboring slice should speed up the solution of the next slice by a good amount; so there is hope of solving even $500 \times 500 \times 500$ real-world problems in time that is acceptable in practice.

The better of the presented algorithms are that good, that it makes sense to compare their *absolute* errors (see Figure 4.22). As can be seen, the absolute error for ImprovementC seems constant for 3 directions. (Of course, it follows from [GGP99] that asymptotically there must be a more than constant worst-case error unless $\mathbb{P} = \mathbb{NP}$.) For four and five directions the absolute error appears to be $O(\sqrt{|G|})$.

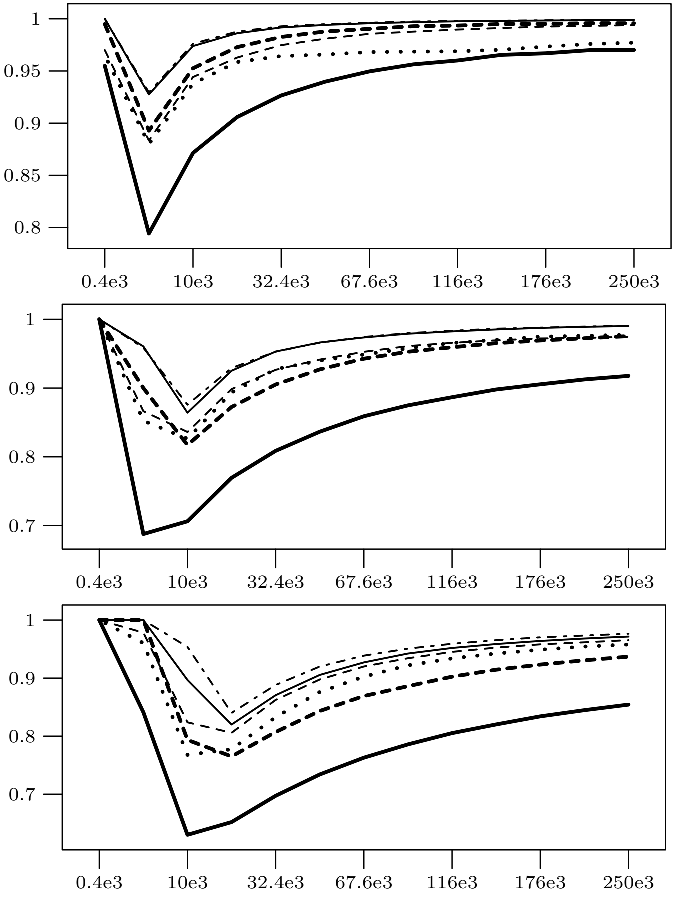


FIGURE 4.9. Relative performance for 3 (*top*), 4 (*middle*), and 5 (*bottom*) directions on instances of 1% density for GreedyA (╱), ImprovementA (╱), GreedyB (·), ImprovementB (╱), GreedyC (╱), and ImprovementC (╱). The abscissa depicts the number of variables at a quadratic scale and the ordinate depicts the relative performance.

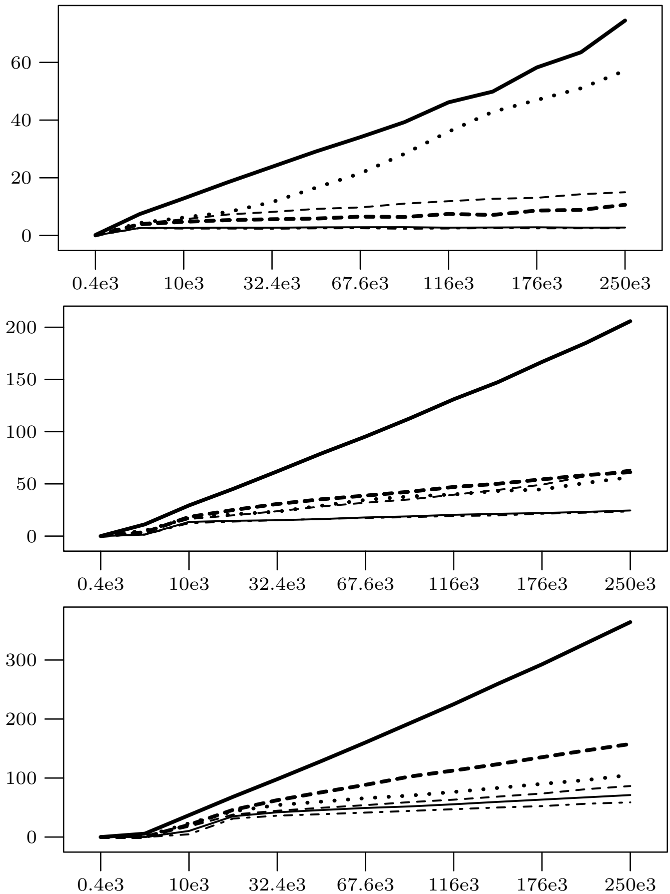


FIGURE 4.10. Absolute error for 3 (*top*), 4 (*middle*), and 5 (*bottom*) directions on instances of 1% density for GreedyA (╱), ImprovementA (╱), GreedyB (·), ImprovementB (╱), GreedyC (╱), and ImprovementC (╱). The abscissa depicts the number of variables at a quadratic scale and the ordinate depicts the absolute error at a **linear(!)** scale.

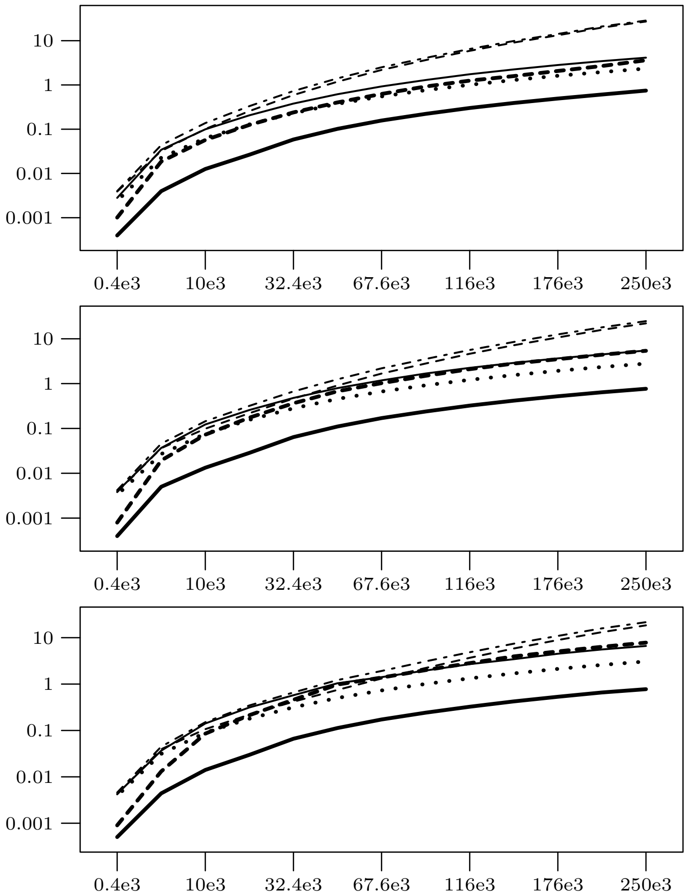


Figure 4.11. Running-times for 3 (*top*), 4 (*middle*), and 5 (*bottom*) directions on instances of 1% density for GreedyA (), ImprovementA (), GreedyB (·), ImprovementB (), GreedyC (), and ImprovementC (). The abscissa depicts the number of variables at a quadratic scale and the ordinate depicts the running times in seconds at a logarithmic scale.

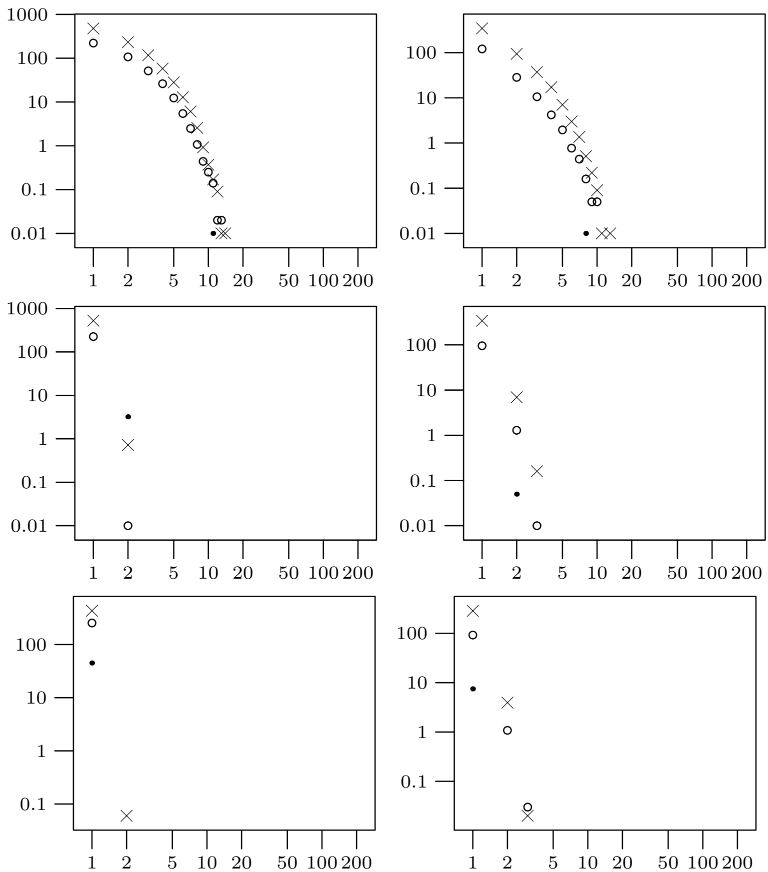


FIGURE 4.12. Distribution of error for 100 instances with 500^2 variables, density 1%, and 3 ($\bullet$), 4 ($\circ$), and 5 ($\times$) directions. Depicted are: GreedyA (*top left*), ImprovementA (*top right*), GreedyB (*middle left*), ImprovementB (*middle right*), GreedyC (*bottom left*), and ImprovementC (*bottom right*). The abscissa depicts the absolute error on a line at a logarithmic scale and the ordinate depicts the average number of lines with this error at a logarithmic scale.

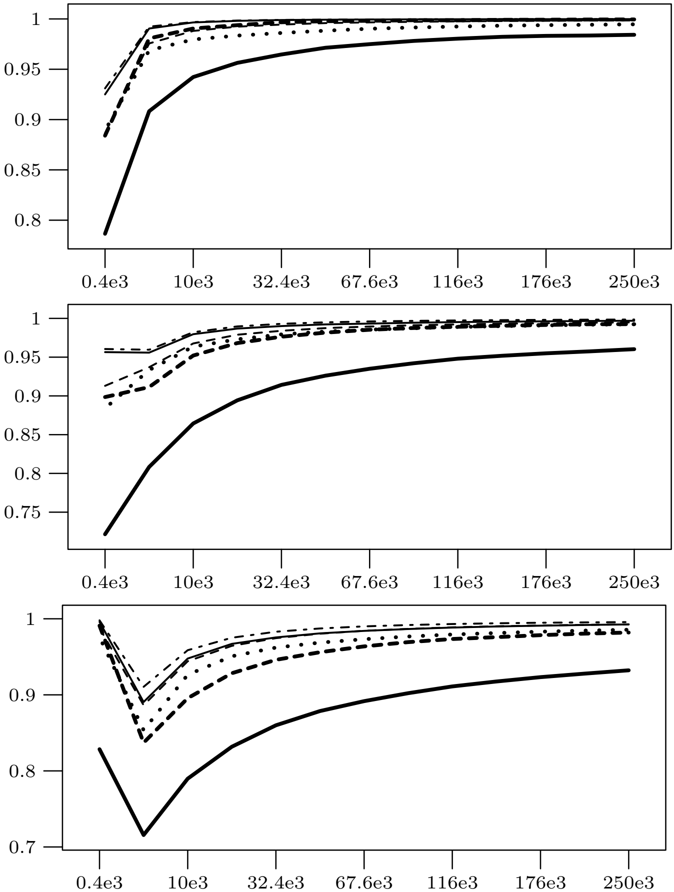


FIGURE 4.13. Relative performance for 3 (*top*), 4 (*middle*), and 5 (*bottom*) directions on instances of 5% density for GreedyA (/), ImprovementA (/), GreedyB (·), ImprovementB (/), GreedyC (/), and ImprovementC (/). The abscissa depicts the number of variables at a quadratic scale and the ordinate depicts the relative performance.

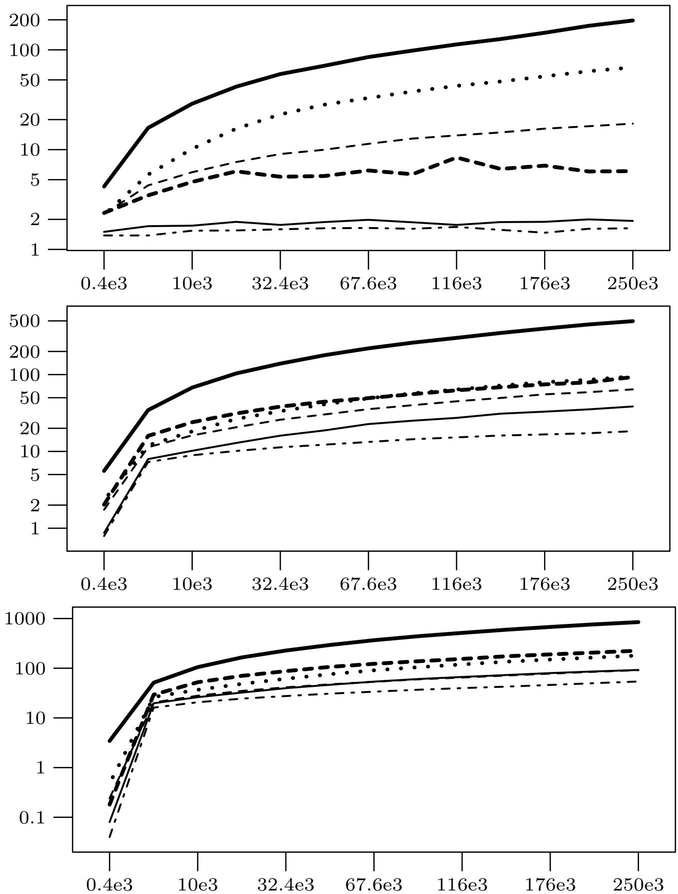


Figure 4.14. Absolute error for 3 (*top*), 4 (*middle*), and 5 (*bottom*) directions on instances of 5% density for GreedyA ( ), ImprovementA ( ), GreedyB (·), ImprovementB ( ), GreedyC ( ), and ImprovementC ( ). The abscissa depicts the number of variables at a quadratic scale and the ordinate depicts the absolute error at a logarithmic scale.

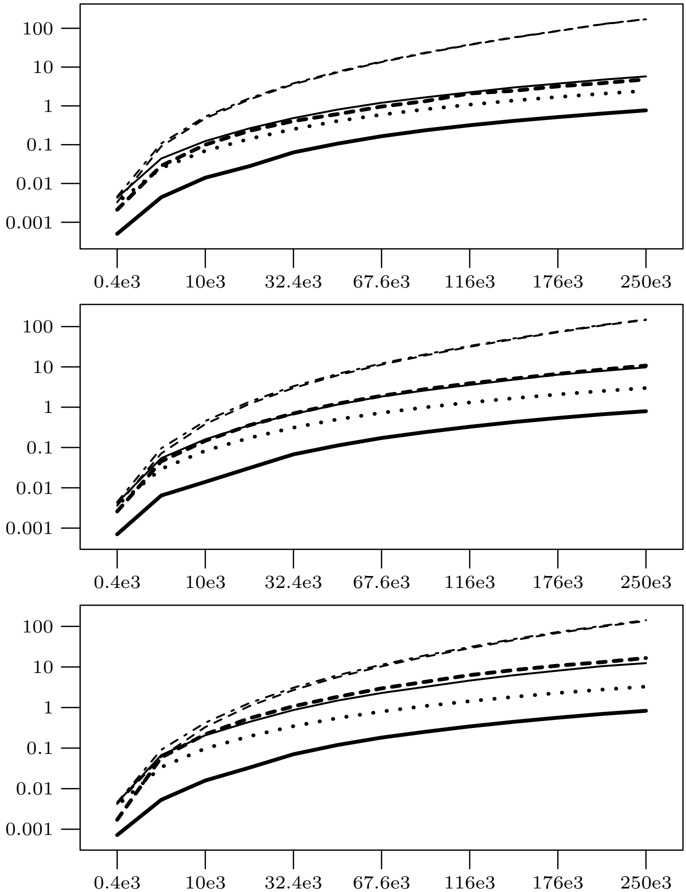


FIGURE 4.15. Running-times for 3 (*top*), 4 (*middle*), and 5 (*bottom*) directions on instances of 5% density for GreedyA (/), ImprovementA (/), GreedyB (·), ImprovementB (/), GreedyC (/), and ImprovementC (/). The abscissa depicts the number of variables at a quadratic scale and the ordinate depicts the running times in seconds at a logarithmic scale.

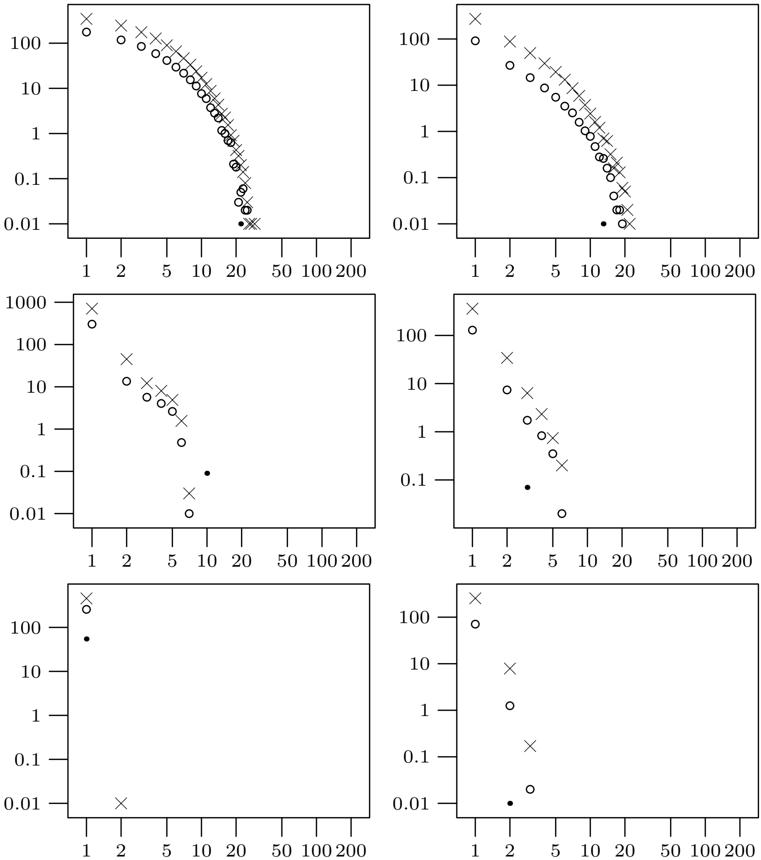


FIGURE 4.16. Distribution of error for 100 instances with 500^2 variables, density 5%, and 3 ($\bullet$), 4 ($\circ$), and 5 ($\times$) directions. Depicted are: GreedyA (*top left*), ImprovementA (*top right*), GreedyB (*middle left*), ImprovementB (*middle right*), GreedyC (*bottom left*), and ImprovementC (*bottom right*). The abscissa depicts the absolute error on a line at a logarithmic scale and the ordinate depicts the average number of lines with this error at a logarithmic scale.

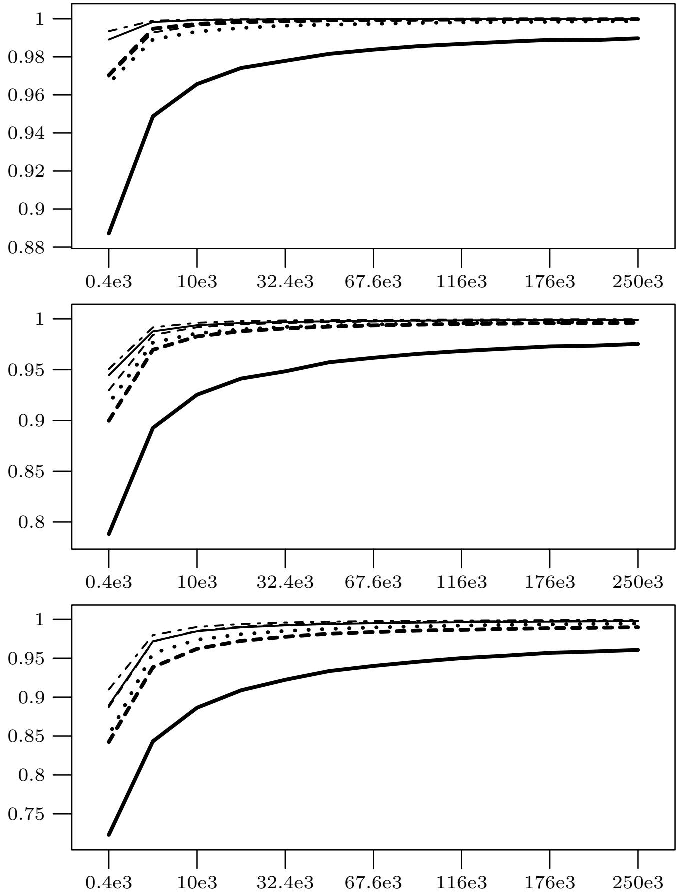


FIGURE 4.17. Relative performance for 3 (*top*), 4 (*middle*), and 5 (*bottom*) directions on instances of 20% density for GreedyA (/), ImprovementA (/), GreedyB (·), ImprovementB (/), GreedyC (/), and ImprovementC (/). The abscissa depicts the number of variables at a quadratic scale and the ordinate depicts the relative performance.

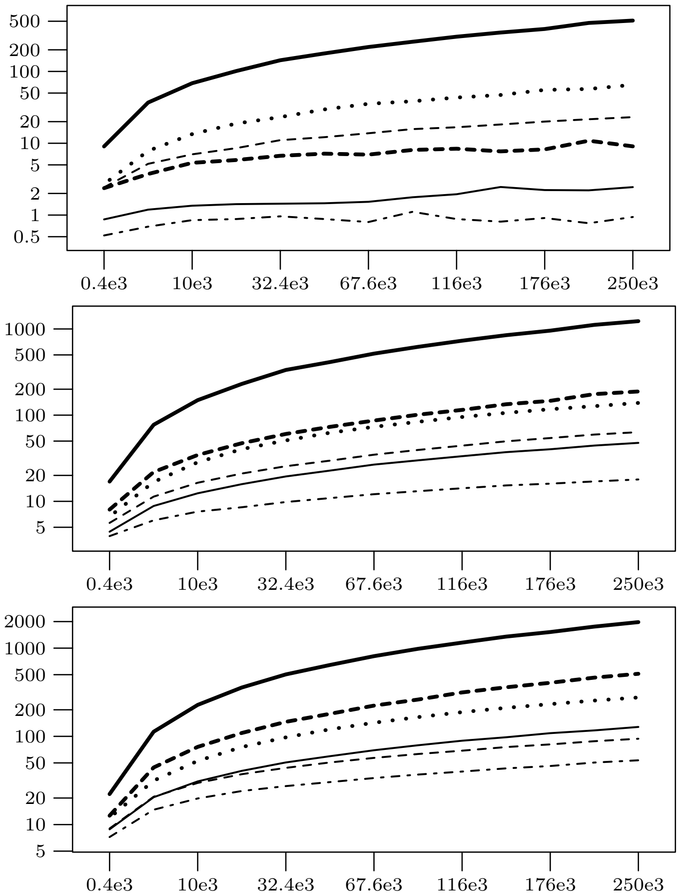


Figure 4.18. Absolute error for 3 (*top*), 4 (*middle*), and 5 (*bottom*) directions on instances of 20% density for GreedyA ( ), ImprovementA ( ), GreedyB ( ), ImprovementB ( ), GreedyC ( ), and ImprovementC ( ). The abscissa depicts the number of variables at a quadratic scale and the ordinate depicts the absolute error at a logarithmic scale.

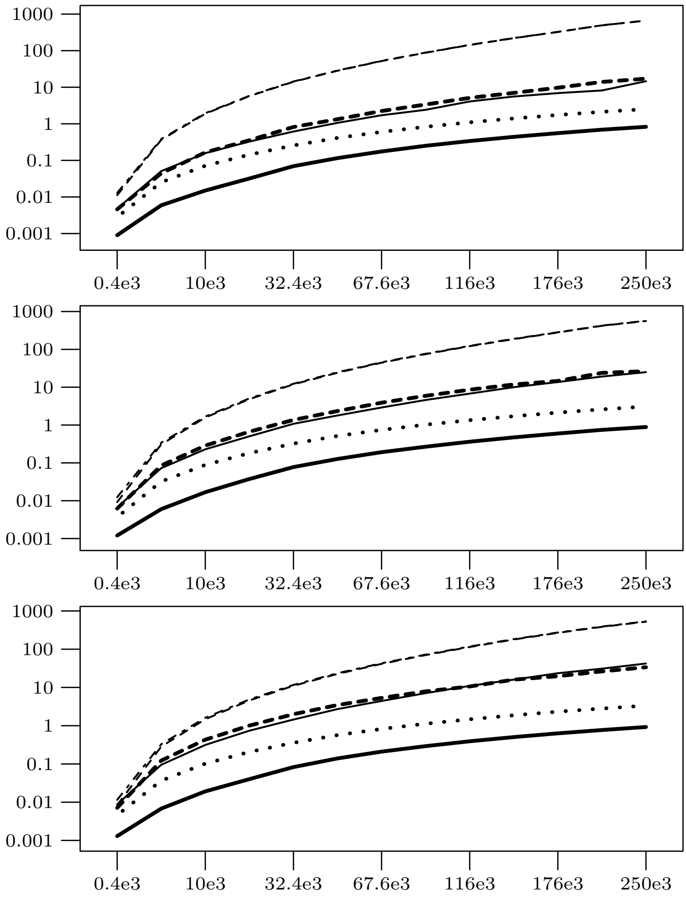


FIGURE 4.19. Running-times for 3 (*top*), 4 (*middle*), and 5 (*bottom*) directions on instances of 20% density for GreedyA (/), ImprovementA (/), GreedyB (·), ImprovementB (/), GreedyC (/), and ImprovementC (/). The abscissa depicts the number of variables at a quadratic scale and the ordinate depicts the running times in seconds at a logarithmic scale.

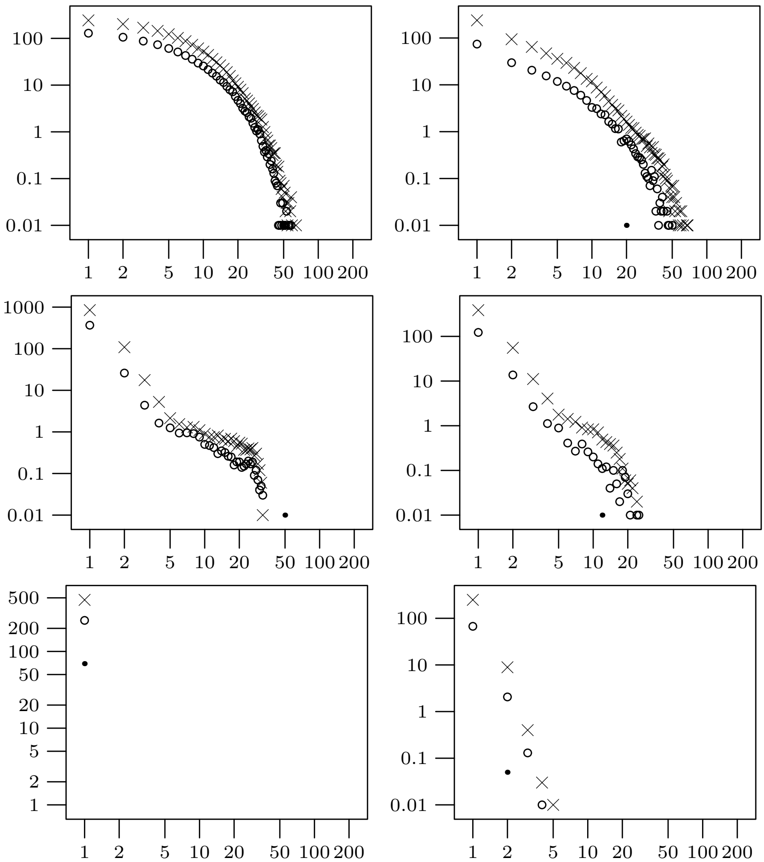


FIGURE 4.20. Distribution of error for 100 instances with 500^2 variables, density 20%, and 3 (•), 4 (∘), and 5 (×) directions. Depicted are: GreedyA (*top left*), ImprovementA (*top right*), GreedyB (*middle left*), ImprovementB (*middle right*), GreedyC (*bottom left*), and ImprovementC (*bottom right*). The abscissa depicts the absolute error on a line at a logarithmic scale and the ordinate depicts the average number of lines with this error at a logarithmic scale.

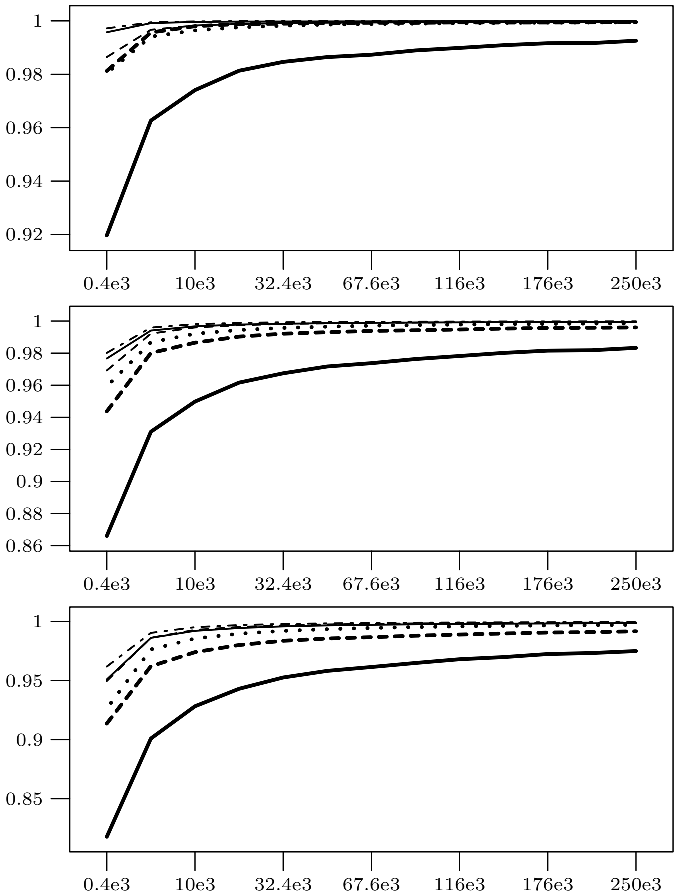


FIGURE 4.21. Relative performance for 3 (*top*), 4 (*middle*), and 5 (*bottom*) directions on instances of 50% density for GreedyA ( ), ImprovementA ( ), GreedyB ( ), ImprovementB ( ), GreedyC ( ), and ImprovementC ( ). The abscissa depicts the number of variables at a quadratic scale and the ordinate depicts the relative performance.

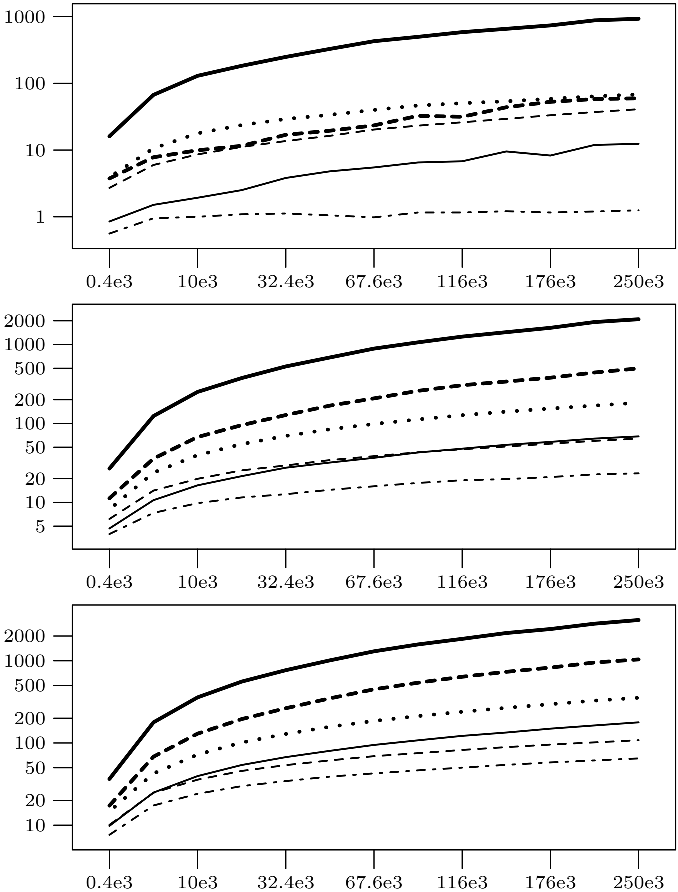


FIGURE 4.22. Absolute error for 3 (*top*), 4 (*middle*), and 5 (*bottom*) directions on instances of 50% density for GreedyA (/), ImprovementA (/), GreedyB (·), ImprovementB (/), GreedyC (/), and ImprovementC (/). The abscissa depicts the number of variables at a quadratic scale and the ordinate depicts the absolute error at a logarithmic scale.

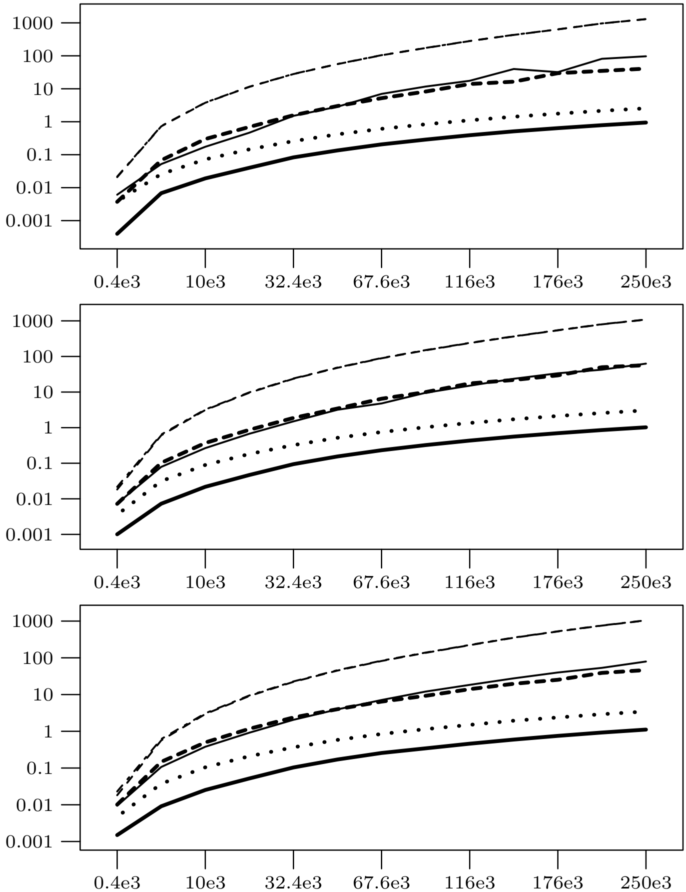


FIGURE 4.23. Running-times for 3 (*top*), 4 (*middle*), and 5 (*bottom*) directions on instances of 50% density for GreedyA (/), ImprovementA (/), GreedyB (·), ImprovementB (/), GreedyC (/), and ImprovementC (/). The abscissa depicts the number of variables at a quadratic scale and the ordinate depicts the running times in seconds at a logarithmic scale.

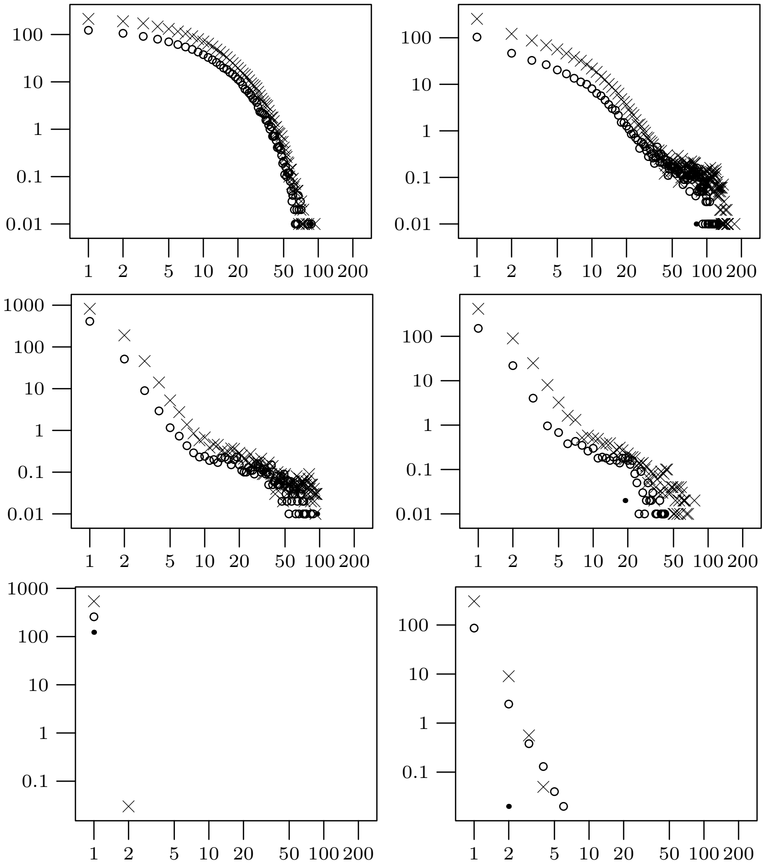


Figure 4.24. Distribution of error for 100 instances with 500^2 variables, density 50%, and 3 ($\bullet$), 4 ($\circ$), and 5 ($\times$) directions. Depicted are: GreedyA (*top left*), ImprovementA (*top right*), GreedyB (*middle left*), ImprovementB (*middle right*), GreedyC (*bottom left*), and ImprovementC (*bottom right*). The abscissa depicts the absolute error on a line at a logarithmic scale and the ordinate depicts the average number of lines with this error at a logarithmic scale.

Another (practically) important issue is that of the distribution of errors among different lines. For this we counted for 100 problems (of size 500×500) how many constraints were satisfied with equality, how many needed only one more point for equality, and so on. Again, it turned out, that the algorithms GreedyC and ImprovementC have the best error distribution. In particular, for GreedyC no line occurred with error greater one for 3 and 4 directions, for 5 directions the worst cases were four instances with a single line of error 2. For ImprovementC the worst cases were two instances with two lines of error 2 for 3 directions, for 4 directions one instance with a single line of error 8 and for 5 directions seven instances with a single line of error 4. In contrast to the simple maximization problem, here the 1-improvements can make a solution worse (with respect to this measure), as it may happen that in a number of improvement steps atoms from the same line are removed.

For GreedyC only lines with error at most 2 occur, while for ImprovementC a single instance with a line of error 8 came up. In contrast, GreedyA, GreedyB, ImprovementA, and ImprovementB have always a couple of lines with a huge error (see Figure 4.24). For instance, for GreedyA, ImprovementA and GreedyB instances occurred with lines of error 66, 122, and 154 for 3 to 5 directions. ImprovementB is better, in that errors occurred only up to 25, 35, and 72. These huge errors do seem inappropriate in the physical application because it is more likely that many lines occur with small error, than some with very large error.

The results for the other densities $1\%, 5\%, 20\%$ are similar except that for very sparse and small problems an artifact crept into the graphs: it turns out that small, sparse instances are exceptionally well solved, because it is easy to see that (for example) problems with two directions and every linesum at most of value one are already solved exactly by the greedy algorithm.

CHAPTER 5

Antiwebs and Antiweb-Wheels for Stable Set Polytopes

5.1. Introduction

Most of the notation we need was already presented in Subsection 2.4.6, here we need only to mention another important class of inequalities, called *antiweb inequalities.* This class contains both cycle and clique inequalities.

Given a class of valid inequalities, $\mathcal{C}$, for STAB(G), and a polyhedron $P \supseteq$ STAB(G) the corresponding *separation problem* is: Given $x^* \in P$, does x^* violate any of the inequalities in $\mathcal{C}$? If the answer is *yes*, exhibit such an inequality. This problem is important if one wants to use the inequalities in a branch-and-cut method to optimize a linear function over STAB(G). See, for examples, Barahona, Weintraub, and Epstein [BWE92] and Nemhauser and Sigismondi [NS92]; for a general branch-and-cut application-framework see Jünger and Thienel [JT98]. Furthermore, if a separation problem is solvable in polynomial time, then the corresponding optimization problem can be solved in polynomial time, see Grötschel, Lovász, and Schrijver [GLS93]. The separation problem for the class consisting of the trivial and edge inequalities with respect to $\mathbb{R}^V_+$ can obviously be solved in $O(m)$ time, but the separation problem for the clique inequalities with respect to ESTAB(G) is $\mathbb{NP}$-hard as proved by Grötschel, Lovász and Schrijver [GLS81].

In contrast, the separation problem for the odd cycle inequalities with respect to ESTAB(G) can be solved in polynomial time. The underlying algorithmic problem of finding minimum weight paths of odd length is first solved by Grötschel and Pulleyblank [GP81], though they "...attribute it to 'Waterloo-folklore' ...". The separation algorithm for the odd cycle inequalities with respect to ESTAB(G) itself is given in [GLS93]. Hence the separation problem for the class consisting of the trivial, edge and cycle inequalities with respect to $\mathbb{R}_+^V$ can be solved in polynomial time. Cheng and Cunningham [CC97] enlarged the class of polynomially separable inequalities by proving that wheel inequalities are separable in polynomial time. As a building block to generalize their result, we shall prove that a generalization of the (t)-antiweb inequalities can be separated in polynomial time (Sec. 5.3), even though the separation of antiweb inequalities is $\mathbb{NP}$-hard (Sec. 5.4). In Section 5.5, we introduce a large new class of valid inequalities, called *antiweb-wheel inequalities,* which are a common generalization of wheel inequalities and antiweb inequalities, and study in Section 5.6 related separation problems. As one prerequisite for our separation algorithms to work is that all clique inequalities of prescribed size are fulfilled, we have to worry about clique separation. One way is to enumerate all cliques of size at most that bound, that is assumed to be fixed. The other—theoretically more appealing way—is to separate the large class of *orthonormal representation cuts* [GLS93, 9.3.2] (here abbreviated by "orthogonality cuts") but it raises the question whether antiweb inequalities might be already implied by the class of orthogonality cuts. In Section 5.8 we show that they are not implied.

We start our study of facetness with Section 5.9 describing three operations—adding an apex, doubly subdividing an edge, and doing star subdivision—that can preserve facetness; furthermore their interaction is studied. The knowledge about these operations is applied in Section 5.10 to give a complete characterization of the facet inducing inequalities among all proper antiweb-1-wheel and antiweb-s-wheel inequalities. Finally, we provide a brief view into questions of facetness for improper antiweb-wheels.

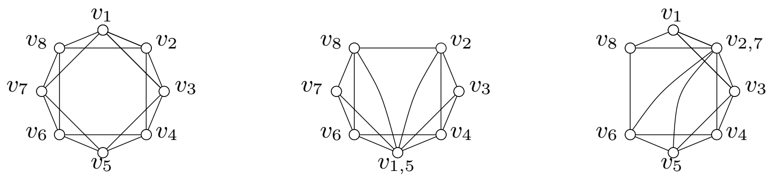
The results of this chapter are joint work with Eddie Cheng.

5.2. Preliminaries

Let n and t be integers such that $t \geq 2$, $n \geq 2t-1$ and $n \not\equiv 0 \pmod{t}$. An (n,t)-*antiweb* $\mathcal{AW}$ is a graph with vertex-set $\{v_1, v_2, \dots, v_n\}$; two vertices v_i and v_j $(i > j)$ are adjacent if $k := \min\{i-j, n+j-i\} \leq t-1$; we call $\{v_i, v_j\}$ a cross-edge of *type* k, or simply k-*edge*. A 1-edge may also be referred to as a *rim edge*. We define the following *distance function* for the vertices of an antiweb with $i > j$ by

$$\operatorname{dist}(v_i, v_j) = \min(i-j, j+n-i).$$

We denote this antiweb by $\mathcal{AW}(v_1, v_2, \dots, v_n)$. (Note that it is an ordered list.) The sequence $(v_1, v_2, \dots, v_n)$ is called the *spine* of the antiweb.



(a) (b) (c)

Figure 5.1. (a) shows a simple $(8, 3)$-antiweb; (b) shows the nonsimple antiweb resulting from identifying vertices v_1 and v_5 in (a); (c) shows the nonsimple antiweb resulting from identifying vertices v_2 and v_7 in (a).

For an example of an $(8, 3)$-antiweb see Figure 5.1(a). The class of $(\cdot, t)$-antiwebs is referred to as (t)-antiwebs. Thus (2)-antiwebs are odd cycles. Our definition is slightly different from the one given in Trotter [Tro75] as we include additionally cliques as antiwebs (in the case $n = 2t - 1$). An (n, t)-antiweb contains n different t-cliques, namely, the t-clique $\mathcal{T}_i$ on the vertices $\{v_i, v_{i+1}, \dots, v_{i+t-1}\}$ for $i = 1, 2, \dots, n$ (where the indices $j > n$ are reduced to $1 + ((j - 1) \bmod n)$). We refer to $\mathcal{T}_1, \mathcal{T}_2, \dots, \mathcal{T}_n$ as the *generators* of the antiweb. The inequality $\sum_{i=1}^{n} x_i \leq \left\lfloor \frac{n}{t} \right\rfloor$ is the *antiweb inequality* described in [Tro75]. If $n = 2t - 1$, then it is a clique inequality. If $t = 2$, then it becomes the cycle inequality. In Euler, Jünger and Reinelt [EJR87] inequalities of some antiwebs (the class of odd anticycles) were generalized to *independence system polytopes;* in Laurent [Lau89], the full class of antiweb inequalities was studied for independence system polytopes. Schulz [Sch96] and Müller and Schulz [MS96] generalize antiwebs further to the general setting of *transitive packing*. The stable set problem for antiwebs is solvable in polynomial time, as they belong to the class of circular arc graphs for which Golumbic and Hammer [GH88] show that stable set is easy. A complete description of the stable set polytope by inequalities is given by Dahl [Dah99] for the case of 3-antiwebs, but neither the question of their separation is treated nor is anything said about (t)-antiweb polytopes for $t > 3$.

We define $\mathrm{A}_t\mathrm{STAB}(G) = \{x \in \mathrm{ESTAB}(G) \colon x$ fulfills the (t)-antiweb inequalities$\}$. In this chapter we will use the term *simple G-configurations* for a simple graph G. Next we consider *nonsimple* G-configurations. Suppose H is obtained from a graph G by a sequence of identifications of *nonadjacent* vertices. Then H is a *nonsimple* G-configuration. If at least one of the identification is between adjacent vertices, then it is a *degenerate* G-configuration. In either case, we assume any duplicate edge is deleted after an identification (but a copy of each loop is kept in the degenerate case). The nonsimple configurations turn out to be more important for the stable set problem under investigation than the degenerate configurations are. From now on, whenever we identify two vertices, we assume implicitly that the two vertices are not adjacent (unless otherwise specified). For example, Figures 5.1(b), 5.1(c) show the nonsimple $(8,3)$-antiwebs $\mathcal{AW}(v_{1,5}, v_2, v_3, v_4, v_{1,5}, v_6, v_7, v_8)$ and $\mathcal{AW}(v_1, v_{2,7}, v_3, v_4, v_5, v_6, v_{2,7}, v_8)$. If G is the support graph of some valid inequality, then a nonsimple G-configuration H also induces the same inequality. (Degenerate G-configurations do not have this property but the concept is useful later on.) The next simple lemma clarifies this.

Lemma 5.2.1.
Let $\sum_{i=1}^{n} a_i x_i \le b$ be a valid inequality for STAB(G) and let v_1 and v_2 be two nonadjacent vertices of G. If H is a nonsimple configuration obtained from G by identifying v_1 and v_2 where the vertex $v_{1,2}$ of H is obtained from the identification of v_1 and v_2 of G, then $(a_1 + a_2)x_{1,2} + \sum_{i=3}^{n} a_i x_i \le b$ is a valid inequality for STAB(H).

PROOF. This follows from the fact that $(x_1^*, x_2^*, x_3^*, \ldots, x_n^*)^T$ (with $x_1^* = x_2^*$) is an incidence vector of a stable set of G whenever $(x_{1,2}^*, x_3^*, \ldots, x_n^*)^T$ is an incidence vector of a stable set of H. □

Let H be a graph and H' be the nonsimple configuration obtained from H by a sequence of identifications of vertices. By applying Lemma 5.2.1 repeatedly, a valid inequality for STAB(H) provides a valid inequality for STAB(H'). Clearly such a sequence of identifications of vertices induces a partition of the vertices of H such that H' is obtained from H by identifying vertices in the same class of the partition. The antiweb inequality for the nonsimple antiweb in Figure 5.1(b) is $x_2 + x_3 + x_4 + x_6 + x_7 + x_8 + 2x_{1,5} \le 2$. (However, this inequality does not induce a facet as it is the sum of $x_6 + x_7 + x_8 + x_{1,5} \le 1$ and $x_{1,5} + x_4 + x_3 + x_2 \le 1$, both facet inducing.) If $\mathcal{AW}$ is a nonsimple (n,t)-antiweb, we still write it as $\sum_{v \in V(\mathcal{AW})} x_v \le \lfloor n/t \rfloor$ and *by convention,* we treat $V(\mathcal{AW})$ as a multiset according to the number of roles a vertex takes in $\mathcal{AW}$ in the corresponding simple configuration. So for our example, $x_{1,5}$ appears twice in $\sum_{v\in V(\mathcal{AW})} x_v$. From now on, the term "class of antiweb inequalities" includes both simple and nonsimple inequalities; for example, Lemma 5.2.3 applies to simple and nonsimple antiweb inequalities. For a nonsimple antiweb inequality written in this form, the following convention is used: Like $V(\mathcal{AW})$, $E(\mathcal{AW})$ is treated as a multiset according to the number of roles an edge takes in $\mathcal{AW}$ as the graph is simple; for instance, if an edge e has two roles in $\mathcal{AW}$, then it appears twice in the summation even though it appears once as an edge in $E(\mathcal{AW})$. For example, $\{v_1, v_{2,7}\}$ in Figure 5.1(c) has the role of a 2-edge once (viewing it as $\{v_1, v_7\}$) and has the role of a 1-edge once (viewing it as $\{v_1, v_2\}$).

In terms of separation, it is desirable to write the required inequalities in a different form. Let $f(v) = -1/4 + x_v/2$ and $w_e = (1 - x_u - x_v)/2$. (So $2w_e$ is the slack for the edge e.) We use f^* and w_e^* to denote $f(v)$ and w_e evaluated at a specific x^*. The next two results give such formulations.

Lemma 5.2.2.
Let I_{K_t} be a t-clique inequality with K_t as its support graph. Then I_{K_t} can be rewritten as

$$(I_{K_t}) \qquad -\sum_{e\in E(K_t)} w_e - (t-3)\sum_{v\in V(K_t)} f(v) + \frac{1}{2}t - 1 \le 0.$$

PROOF. We consider the coefficient of each term. Consider the multivariable function

$$\phi = -\sum_{e\in E(K_t)} w_e - (t-3)\sum_{v\in V(K_t)} f(v) + \frac{1}{2}t - 1$$

and let $[p]\psi$ denote the coefficient of the indeterminate p in ψ and $[1]\psi$ denotes the constant term for any function ψ.

1. $[x_v]\phi$ *where* $v \in V(K_t)$*:* Then $[x_v](\sum_{e\in E(K_t)} w_e) = -(t-1)/2$ and $[x_v]f(v) = 1/2$. Hence $[x_v]\phi = (t-1)/2 - (t-3)/2 = 1$.
2. $[1]\phi$*:* We note that $[1](\sum_{e\in E(K_t)} w_e)$ is equal to $1/2$ times the number of edges in K_t. The number of edges in the clique is $t(t-1)/2$. Moreover, we observe that $[1]f(v) = -1/4$. Hence $[1]\phi = -(t^2 - t)/4 + t(t-3)/4 + t/2 - 1 = -1$. □

Lemma 5.2.3.
Let $I_{\mathcal{AW}}$ be an (n,t)-antiweb inequality with $\mathcal{AW}$ as its support graph. Then $I_{\mathcal{AW}} : \sum_{v\in V(\mathcal{AW})} x_v \leq \lfloor n/t \rfloor$ can be rewritten as

$$(I_{\mathcal{AW}}) \qquad - \sum_{e\in E(\mathcal{AW})} w_e - (2t-4) \sum_{v\in V(\mathcal{AW})} f(v) + \frac{n}{2} - \left\lfloor \frac{n}{t} \right\rfloor \leq 0.$$

PROOF. It is enough to prove this for $\mathcal{AW}$ being simple. The nonsimple part follows immediately from the simple case. We consider the coefficient of each term. Consider the multivariable function

$$\phi = - \sum_{e\in E(\mathcal{AW})} w_e - (2t-4) \sum_{v\in V(\mathcal{AW})} f(v) + \frac{n}{2} - \left\lfloor \frac{n}{t} \right\rfloor.$$

1. $[x_v]\phi$ *where* $v \in V(\mathcal{AW})$*:* Then follows $[x_v](\sum_{e\in E(\mathcal{AW})} w_e) = -(2t-2)/2$ and $[x_v]f(v) = 1/2$. Hence $[x_v]\phi = (2t-2)/2 - (2t-4)/2 = 1$.
2. $[1]\phi$*:* We note that $[1](\sum_{e\in E(\mathcal{AW})} w_e)$ is equal to $1/2$ times the number of edges in $\mathcal{AW}$. The number of edges in the antiweb is $n(t-1)$. Moreover, we observe that $[1]f(v) = -1/4$. Hence $[1]\phi = -(nt-n)/2 + n(2t-4)/4 + n/2 - \lfloor n/t \rfloor = -\lfloor n/t \rfloor$. □

5.3. Finding a Minimum-Weight (t)-Antiweb

In this section, we give a polynomial time algorithm for finding a minimum weight (t)-antiweb (possibly degenerate) in a given graph and apply it to the problem of (t)-antiweb separation. In contrast to this we show in Section 5.4 that recognizing an antiweb in a graph is $\mathbb{NP}$-complete; thereby separation of antiwebs is $\mathbb{NP}$-hard.

The minimum weight (t)-antiweb algorithm is used as a subroutine by all other separation algorithms in this chapter. We note that the graph in MINIMUM-WEIGHT-t-ANTIWEB (defined below) might have loops. This is not an inconsistency as the graph in MINIMUM-WEIGHT-t-ANTIWEB may not be the one in which we want to find a stable set. It is just a graph in which we are looking for a minimum weight configuration. Because this graph may have loops, it may contain degenerate cliques, that is, a complete graph with loops at some of its vertices. For example, the graph with two vertices a and b with edge $\{a,b\}$ and a loop at a can be considered a degenerate 3-clique, and a loop at a vertex can be considered a degenerate t-clique for any $t \geq 2$. It may also contain degenerate antiwebs. An *ordered t-clique* is a clique on $v_1, v_2, \ldots, v_t$ (not necessarily distinct) in which the order of the vertices is important.

MINIMUM-WEIGHT-t-ANTIWEB.

Instance: *Numbers* $1 \leq q \leq t-1$, $r \geq 2t-1$, *a graph* $G = (V, E)$ *(possibly with loops), and for each ordered* t*-clique* $\mathcal{T}$ *(possibly degenerate), a nonnegative weight* $w_G(\mathcal{T})$.

Output: *A* t*-antiweb (possibly degenerate) with minimum weight among all* (n,t)*-antiwebs with* $n \equiv q \bmod t$ *and* $n \geq r$. *Here the weight of an* (n,t)*-antiweb is the sum of the weights of its generators, each ordered naturally*

We note that by "naturally" we mean the weight of $\mathcal{AW}(v_1, v_2, \dots, v_n)$ is the weight of the following ordered t-cliques: $(v_1, v_2, \dots, v_t), (v_2, v_3, \dots, v_{t+1}), \dots, (v_n, v_1, \dots, v_{t-1})$. For example, for the clique with vertices $v_1, v_2, \dots, v_t$, we must use the weight for the ordered t-clique $(v_1, v_2, \dots, v_t)$ and not $(v_2, \dots, v_t, v_1)$. (In the special case that $t = 2$, $q = 1$ and $r = 3$, Problem MINIMUM-WEIGHT-t-ANTIWEB is the task of finding a minimum-weight odd closed walk, for which the solution is well-known, see [GLS93].) Next we construct a directed graph D from G. The vertex-set V_D of D is constructed as follows: There is a vertex in V_D for every ordered t-clique $\mathcal{T}$ (possibly degenerate), with ordering $(u_1, u_2, \dots, u_t)$, in G. Notice that for a fixed t, $|V_D| = O(|V|^t)$ holds. The arcs in D are constructed as follows: there is an arc from $(u_1, u_2, \dots, u_t)$ to $(v_1, v_2, \dots, v_t)$ if and only if $u_i = v_{i-1}$ for $i = 2, 3, \dots, t$; moreover, the weight on this arc is the weight of the ordered t-clique $\mathcal{T}(u_1, u_2, \dots, u_t)$. Let $\mathcal{AW}(v_1, v_2, \dots, v_n)$ be an (n,t)-antiweb (possibly degenerate) in G. Construct the following directed closed walk C of length n in D:

$$(v_1, v_2, \dots, v_t) \to (v_2, v_3, \dots, v_{t+1}) \to \cdots \to (v_n, v_1, \dots, v_{t-1}) \\ \to (v_1, v_2, \dots, v_t).$$

Now it follows from construction that $w_G(\mathcal{AW}) = w_D(C)$. Conversely, given a directed closed walk

$$(u_1, u_2, \dots, u_t) \to (u_2, u_3, \dots, u_{t+1}) \to \cdots \to (u_n, u_1, \dots, u_{t-1}) \\ \to (u_1, u_2, \dots, u_t)$$

of length n in D, then by construction, $(u_i, u_{i+1}, \dots, u_{i+t-1})$ gives an ordered t-clique (possibly degenerate) in G where $1 \leq i \leq n$; let this be $\mathcal{T}_i$. Then $\mathcal{T}_1, \mathcal{T}_2, \dots, \mathcal{T}_n$ generate an (n,t)-antiweb $\mathcal{AW}$ with $w_G(\mathcal{AW}) =$

$w_D(C)$. Moreover, this $\mathcal{AW}$ is, in fact, a subgraph of G. In particular, if G has no loops, then $\mathcal{AW}$ can be nonsimple but not degenerate. Hence problem MINIMUM-WEIGHT-t-ANTIWEB is equivalent to finding a minimum weight directed closed walk of length n in D with $n \equiv q \pmod t$ and $n \geq r$. This is just a REMAINDER-RESTRICTED-ALL-PAIRS-SHORTEST-WALK problem, which can be solved in polynomial time, see Theorem 2.4.1 in Subsection 2.4.3. This gives the following result.

Theorem 5.3.1.
MINIMUM-WEIGHT-t-ANTIWEB *can be solved in polynomial time.*

If $t = 2$ then the problem is only to find an odd cycle so in this case we can avoid the detour via the directed graph D and instead apply REMAINDER-RESTRICTED-ALL-PAIRS-SHORTEST-WALK directly. This corresponds to the algorithm given in [GLS93].

Let $\mathrm{Q}_t\mathrm{STAB}(G) = \{x \in \mathrm{ESTAB}(G)\colon x$ fulfills all clique inequalities for cliques of size at most $t\}$. Theorem 5.3.1 will be used to solve various separation problems. First we apply it to the (t)-antiweb inequalities.

Theorem 5.3.2.
Let $t \geq 2$ be fixed. The class of (t)-antiweb inequalities can be separated with respect to $Q_tSTAB(G)$ in polynomial time.

PROOF. Because t is fixed, the class of trivial inequalities and clique inequalities of size t can be separated in polynomial time. Suppose x^* belongs to $\mathrm{Q}_t\mathrm{STAB}(G)$ where G is a simple graph. Because t is fixed, it is enough to give a method to separate all the (n,t)-antiweb inequalities with $n \equiv q \pmod t$ for a fixed q where $1 \leq q \leq t-1$ and $n \geq 2t-1$. Recall that the (n,t)-antiweb inequality for $\mathcal{AW}$ can be written as

$$-\sum_{e \in E(\mathcal{AW})} w_e - (2t-4)\sum_{v \in V(\mathcal{AW})} f(v) + \left(\frac{n}{2} - \left\lfloor \frac{n}{t} \right\rfloor\right) \leq 0.$$

Now we assign to each ordered t-clique $\mathcal{T}(u_1, u_2, \ldots, u_t)$ in G (such clique is not degenerate as G has no loops, but it may be nonsimple) the weight

$$\frac{1}{t}\left(\sum_{1 \leq j < i \leq t} w^*_{u_i,u_j} + (t-3)\sum_{i=1}^{t} f^*(u_i) - \frac{1}{2}t + 1\right).$$

It is nonnegative because the term is just a multiple of a K_t-inequality and x^* satisfies all of them. By definition, $w_{u_i,u_j} + f(u_i) + f(u_j) = 0$. Hence this weight can be rewritten as

$$\sum_{1\le j<i\le t}\left(\frac{1}{t-(i-j)}\right)w^*_{u_i,u_j}+\frac{t-3}{t}\sum_{i=1}^{t}f^*(u_i)$$
$$+\sum_{1\le j<i\le t}\left(\frac{1}{t-(i-j)}-\frac{1}{t}\right)(f^*(u_i)+f^*(u_j))-\frac{1}{2}+\frac{1}{t}.$$

Suppose that ordered t-tuples, $(u^1_1,u^1_2,\ldots,u^1_t),(u^2_1,u^2_2,\ldots,u^2_t),\ldots,(u^n_1,u^n_2,\ldots,u^n_t)$ are given such that $u^i_{j+1}=u^{i+1}_j$ for $i=1,2,\ldots,n-1$ and $j=1,2,\ldots,t-1$, and $u^n_j=u^1_{j+1}$ for $j=1,2,\ldots,t-1$. The ordered t-cliques corresponding to these n ordered t-tuples form the (n,t)-antiweb $\mathcal{AW}(v_1,v_2,\ldots,v_n)$. Moreover, the total weight of them is

$$\sum_{e\in E(\mathcal{AW})}w^*_e+(2t-4)\sum_{v\in V(\mathcal{AW})}f^*(v)+\frac{n}{t}-\frac{n}{2}.$$

Note that this is almost (except for the constant term) the left hand side of an antiweb inequality as written in Lemma 5.2.3. To see this, we first observe that we only have to prove the claim in which the resulting (n,t)-antiweb is simple; the nonsimple case follows immediately (by using Lemma 5.2.1). Consider:

1. *Terms involving* w^*: Consider v_i and v_j, with $j<i$, where $\{v_i,v_j\}$ is a k-edge in the generated (n,t)-antiweb. Then $w^*_{v_i,v_j}$ appears in $(t-k)$ of the t-tuples; this gives $w^*_{v_i,v_j}$ for the total contribution as $k=i-j$.
2. *Terms involving* f^*: Let $1\le k\le n$. Then v_k appears in exactly t of the n given t-tuples. Moreover, it appears as the j-th entry of a t-tuple exactly once for every $1\le j\le t$. Hence the total contribution is (by utilizing $\sum_{1\le j<i\le t}\frac{1}{t-(i-j)}=t-1$)

$$\left(\frac{t-3}{t}\sum_{i=1}^{t}1+2\sum_{1\le j<i\le t}\left(\frac{1}{t-(i-j)}-\frac{1}{t}\right)\right)f^*(v_i)=(2t-4)f^*(v_i).$$

3. *The rest of the terms:* Because the "constant term" in each of the n given t-tuples is $-\frac{1}{2}+\frac{1}{t}$ the total contribution is $-\frac{n}{2}+\frac{n}{t}$.

We apply Theorem 5.3.1 for each $1\le q\le t-1$ with $r=2t-1$ to find a minimum-weight (t)-antiweb in G and its weight. This value is less than the constant $\left(\frac{n}{t}-\left\lfloor\frac{n}{t}\right\rfloor\right)=\frac{q}{t}$ whenever there is a violated (t)-antiweb inequality. We note that we do not have to worry about such a (t)-antiweb

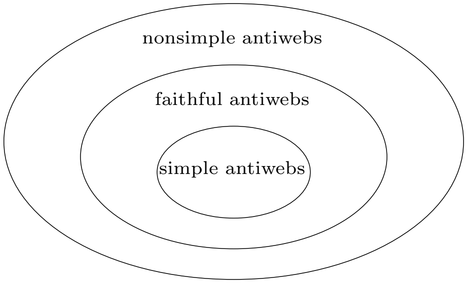


FIGURE 5.2. Relation among the different generalizations of antiwebs.

being degenerate as G has no loops. However, it may be a nonsimple (t)-antiweb. Moreover, this minimum-weight (t)-antiweb gives a most-violated inequality for the corresponding q. (We note that as an antiweb produced by this algorithm may be nonsimple, the inclusion of nonsimple antiweb inequalities is crucial for the validity of our separation algorithm.) $\square$

5.4. Intractability of Separation and Recognition

After we settled the polynomial time separability of the (t)-antiweb inequalities in Section 5.3 one might wonder why we did not aim at the separation of antiwebs in general but considered only the (t)-antiwebs. The reason for this—proved in the current section—is that the general separation problem for antiweb inequalities is $\mathbb{NP}$-hard.

Definition 5.4.1 (Faithful Antiweb)**.**
A nonsimple (n,t)-antiweb $\mathcal{AW}(v_1, v_2, \ldots, v_n)$ is faithful *if $v_i \neq v_{i+j}$ for all indices $i = 1, 2, \ldots, n$ and $0 < j \leq t$.*

Figure 5.2 shows the relation between the different generalizations of antiwebs. Now we can present the four corresponding separation and recognition problems.

SIMPLE-ANTIWEB-RECOGNITION.

Instance: *A graph G and numbers $J, r \in \mathbb{N}$.*

Output: *A simple (n, J)-antiweb with $n \equiv r \bmod J$ contained in G or if there is no such simple (n, J)-antiweb the answer* no.

faithful-Antiweb-Recognition.

Instance: *A graph G and numbers $J, r \in \mathbb{N}$.*

Output: *A faithful (n, J)-antiweb with $n \equiv r \bmod J$ contained in G or if there is no such faithful (n, J)-antiweb the answer* no.

simple-Antiweb-Separation.

Instance: *A graph G, $x^* \in ESTAB(G)$, and numbers $J, r \in \mathbb{N}$.*

Output: *A simple (n, J)-antiweb inequality that is violated by x^*, with $n \equiv r \bmod J$ or if no such inequality exists the answer* no.

faitful-Antiweb-Separation.

Instance: *A graph G, $x^* \in ESTAB(G)$, and numbers $J, r \in \mathbb{N}$.*

Output: *A faithful (n, J)-antiweb inequality that is violated by x^*, with $n \equiv r \bmod J$ or if no such inequality exists the answer* no.

We will start the complexity study by constructing a problem equivalent to the recognition problems. Notice that in the definition of arcs in $\mathcal{G}_t$ in the next theorem we use only those arcs of the digraph D (from the proof of Theorem 5.3.1) that fulfill additionally $c_1 \neq d_t$.

Theorem 5.4.2.
Given a graph $G(V, E)$ define the digraph $\mathcal{G}_t(\mathcal{V}_t, \mathcal{E}_t)$, with

$$\mathcal{V}_t = \{(c_1, c_2, \ldots, c_t) \colon \{c_1, c_2, \ldots, c_t\} \text{ is a } t\text{-clique in } G\}$$

and a pair $((c_1, c_2, \ldots, c_t), (d_1, d_2, \ldots, d_t))$ forms an arc of $\mathcal{G}_t$ if $(c_2, \ldots, c_t) = (d_1, \ldots, d_{t-1})$ and $c_1 \neq d_t$, The graph $G(V, E)$ contains a faithful (n, t)-antiweb if and only if $\mathcal{G}_t(\mathcal{V}_t, \mathcal{E}_t)$ contains a directed n-circuit. Similar, if a *graph $G(V,E)$ contains a simple (n,t)-antiweb then the digraph $\mathcal{G}_t(\mathcal{V}_t,\mathcal{E}_t)$ contains a directed n-circuit.*

PROOF. Clearly, if G contains a faithful (n,t)-antiweb with spine $(v_1,v_2,\dots,v_n)$, then follows that the tuples $(v_i,\dots,v_{i+t-1})$ belong to $\mathcal{V}_t$ for all i and that there are arcs $((v_i,\dots,v_{i+t-1}),(v_{i+1},\dots,v_{i+t}))$ for all i. Hence $\mathcal{G}_t$ contains a directed n-circuit.

If on the other hand $\mathcal{G}_t$ contains a directed n-circuit

$$((v_1,\dots,v_t),(v_2,\dots,v_{t+1}),\dots,(v_{n-1},v_n,\dots,v_{t-2}),(v_n,v_1\dots,v_{t-1}))$$

then the sequence $(v_1,v_2,\dots,v_n)$ is the spine of a faithful (n,t)-antiweb in G. □

Theorem 5.4.3.
For every vertex of $\mathcal{G}_t$ of an (n,t)-antiweb the in- and outdegree coincide and are 0 *or* 1.

PROOF. Consider a vertex $(w_2,\dots,w_{t+1})$ of $\mathcal{G}_t$. Suppose, that the indegree of this vertex is at least two. Then two different vertices $(w_1',w_2,\dots,w_t)$ and $(w_1'',w_2,\dots,w_t)$ exist in $\mathcal{G}_t$. Because all three sequences are vertices of $\mathcal{G}_t$ we know that their underlying sets form t-cliques of $\mathcal{AW}(n,t)$. Furthermore, w_{t+1},w_1',w_1'' are adjacent in G to all vertices in $\{w_2,w_3,\dots,w_t\}$. But we know that in G only two such vertices v_1,v_{t+1} are adjacent to $\{w_2,w_3,\dots,w_t\}$. The condition, that $(w_1',\dots,w_t)$ and $(w_1'',\dots,w_t)$ are predecessors of $(w_2,\dots,w_{t+1})$ in $\mathcal{G}_t$ ensures that $w_1',w_1''\neq w_{t+1}$. Hence $w_1'=w_1''$, contradicting the assumption.

The case of the outdegree can be handled by applying a similar reasoning to the (isomorphic) image of $\mathcal{G}_t$ under the map $v_i\mapsto v_{n+1-i}$.

The fact that $(w_2,\dots,w_{t+1})$ has indegree 1 establishes the existence of a predecessor $(w_1,\dots,w_t)$ in $\mathcal{G}_t$. This together shows, that there is a clique $(v_i,\dots,v_{i+t-1})$ (or $(v_{i+t-1},v_{i+t-2},\dots,v_i)$) in G, underlying $(w_2,\dots,w_{t+1})$. This shows that there is another arc in $\mathcal{G}_t$ from $(w_2,\dots,w_{t+1})$ to $(w_3,\dots,w_{t+1},v_{i+t})$ (or $(w_3,\dots,w_{t+1},v_0)$). So the outdegree is also at least 1. □

Theorem 5.4.4.
The graph $\mathcal{G}_t$ of a simple (n,t)-antiweb G contains two disjoint directed n cycles and $(t!-2)*n$ isolated vertices.

Proof. The two disjoint cycles stem from G. The number of vertices is also clearly $t!n$. (As there are n cliques of size t in G and there are $t!$ different ways to order them.) It remains to show that there are no other edges.

Notice that every edge corresponds to an ordering T of a $(t-1)$-clique of G and two different vertices v, w of $G \setminus T$ so that (v, T) and (T, w) are vertices of $\mathcal{G}_t$ (as the underlying sets are t-cliques of G). Now we distinguish two cases depending on whether the underlying sets of (v, T) and (T, w) are different or if they are equal.

If they are different then T is a $(t-1)$-clique contained in two different t-cliques. But this is only possible if the vertices in T are ordered like an interval on the rim of G. Furthermore, v and w have to be adjacent (on the rim) to different ends of T. Hence the arc belongs to one of the two dicycles.

If the underlying sets are equal, this requires $v = w$. But this is impossible for an edge. So all arcs of $\mathcal{G}_t$ belong to one of the two n-dicycles. ☐

A simple consequence is that no simple (n, t)-antiweb contains another (n', t)-antiweb with $n' < n$. But a simple (n, t)-antiweb contains for all $k \geq 1$ nonsimple (kn, t)-antiwebs.

We shall start the intractability results by proving that simple-Antiweb-Recognition and faithful-Antiweb-Recognition are $\mathbb{NP}$-hard. This will then be used to prove that simple-Antiweb-Separation and faitful-Antiweb-Separation are $\mathbb{NP}$-hard.

Theorem 5.4.5.
The problems simple-Antiweb-Recognition *and* faithful-Antiweb-Recognition *are* $\mathbb{NP}$*-hard.*

Proof. Obviously, both recognition problems belong to $\mathbb{NP}$, as it is a simple task to check, whether a given sequence of vertices constitutes the spine of a faithful antiweb or not.

We will show $\mathbb{NP}$-hardness for the recognition problems by reducing the Clique-problem, see Section 2.2, to them. The Clique-problem is $\mathbb{NP}$-complete [Kar72]. Given an instance (G, J) of the clique problem we will construct a graph G' and a number J' $(= J+1)$ so that the clique problem can be solved by solving the instance (G', J', r) of simple-Antiweb-Recognition.

For G' we start with a $(3J'+r+2, J')$-antiweb, remove all edges between neighbors of a given vertex v, which are on 'different' sides of v, delete the vertex v and make all neighbors of v adjacent to all vertices in G; we call the new graph G'. By 'neighbors of v on different sides of v' we mean neighbors of v so that the shortest path between them that uses only rim-edges contains v (that is neighbors u, w of v with $\operatorname{dist}(u, v) + \operatorname{dist}(v, w) = \operatorname{dist}(u, w)$). There are two reasons for the choice of the number $3J'+r+2$:

1. We remove one vertex and hope then for a $J' - 1$ clique to fill the gap, this makes a total change of $J' - 2$; so the size of the antiweb becomes $3J' + r + 2 + J' - 2$ and this is congruent to r mod J'.
2. The coefficient 3 in $3J' + ...$ is important to ensure that the notion of different sides behaves properly.

Now, if G' contains an (n, J')-antiweb with $n \equiv r \bmod J'$, this antiweb either does not make use of the antiweb we started with, which means that G surely contains a J'-clique (and hence a J-clique); or the discovered antiweb uses the constituents of the old antiweb (which is no longer an antiweb in G' due to the removed vertex) plus J vertices of G which form a J-clique.

To see the assertion, note the following. Suppose G contains no J-clique, but G' contains an antiweb; now have a look at $\mathcal{G}_{J'}(G)$. From the antiweb we started with, we know of two dipaths in $\mathcal{G}_{J'}$. By the assumption that G contains no cliques larger then $J - 1$ we know that all new nodes in $\mathcal{G}_{J'}$ have to contain two vertices from the $\mathcal{AW}$.

Case 1: If there were a new path in $\mathcal{G}$ connecting the two ends of the old path then there has to be a vertex W in $\mathcal{G}$ containing the two immediate neighbors of v in the old antiweb plus $J - 1$ vertices from G. But the two neighbors are not connected, so W cannot be a clique. Contradiction!

Case 2: If there were a new path in $\mathcal{G}$ connecting the head of the one old path with the tail of the other old path there has to be a vertex W in $\mathcal{G}$ containing an immediate neighbor of v in the old antiweb twice plus $J - 1$ vertices from G. But this is again impossible. Contradiction!

For the opposite direction notice, that if G contains a J-clique, then G' contains a J'-antiweb.

Notice, that we required nowhere the involved antiweb to be simple, so our proof is independent of the distinction simple versus faithful. $\square$

Theorem 5.4.6.
The problems SIMPLE-ANTIWEB-SEPARATION *and* FAITFUL-ANTIWEB-SEPARATION *are* $\mathbb{NP}$*-hard.*

PROOF. Given an instance (G, J, r) of of the ANTIWEB-RECOGNITION problem we construct an LP-solution $\hat{x} \in \text{ESTAB}(G)$ (actually, $\hat{x} \in \text{Q}_J\text{STAB}(G)$) by setting all components to $\frac{1}{J}$. Clearly, this vector belongs to $\text{ESTAB}(G)$ (for $|J| \geq 2$).

If the corresponding ANTIWEB-SEPARATION finds for $(G, \hat{x}, J, r)$ a violated (n,t)-antiweb inequality with $n \equiv r \bmod J$ we are happy. If by contrast G contains a faithful/simple (n,t)-antiweb with $n \equiv r \bmod J$, then it is easy to see, that $\hat{x}$ violates the corresponding generalized antiweb inequality. □

5.5. Antiweb-1-Wheel Inequalities

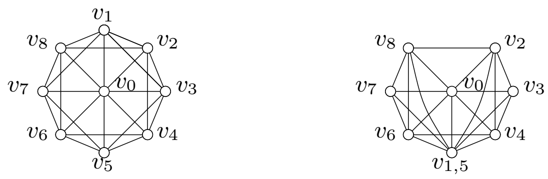
Given a graph $G(V,E)$ and a vertex $v_0 \notin V$, we define the graph G^{v_0} on the vertex set $V \cup \{v_0\}$ and edge set $E \cup \{\{u, v_0\} \colon u \in V\}$.

Definition 5.5.1 (simple antiweb-1-wheel)**.**
Given an (n,t)-antiweb $G_1 = (V_1, E_1)$ with vertex set $V_1 = \{v_1, v_2, \ldots, v_n\}$. Partition V_1 into $\mathcal{E}$ and $\mathcal{O}$. Consider a subdivision G of $G_1^{v_0}$. Let $P_{0,i}$ denote the path obtained from subdividing the edge $\{v_0, v_i\}$ (called a spoke*), and let $P_{i,j}$ (for v_i, v_j adjacent in G_1) denote the path obtained from subdividing the edge $\{v_i, v_j\}$. This resulting graph $\mathcal{AWW}$ is a* simple antiweb-1-wheel *if it satisfies the following conditions:*

1. *The length of $P_{0,i}$ is even for $i \in \mathcal{E}$ and odd for $i \in \mathcal{O}$;*
2. *the length of the path $P_{i,j}$ is even for $i \in \mathcal{E}$ and $j \in \mathcal{O}$ or $j \in \mathcal{E}$ and $i \in \mathcal{O}$;*
3. *the length of the path $P_{i,j}$ is odd for $i,j \in \mathcal{O}$; and*
4. *the length of the path $P_{i,j}$ is odd for $i,j \in \mathcal{E}$.*

Moreover let $\mathcal{S}(\mathcal{AWW})$, or simply $\mathcal{S}$, be the set of internal vertices of $P_{0,i}$ for $i = 1, 2, \ldots, n$, and $\mathcal{R}(\mathcal{AWW})$, or simply $\mathcal{R}$, be the set of internal vertices of all the $P_{i,j}$'s for all $i,j \in \{1, 2, \ldots, n\}$. The vertex v_0 is the hub *of the antiweb-1-wheel, and the vertices in $\mathcal{E} \cup \mathcal{O}$ are the* spoke-ends.

See Figure 5.3(a) for a simple antiweb-1-wheel and Figure 5.3(b) for a nonsimple antiweb-1-wheel. Figure 5.4 depicts a simple antiweb-1-wheel with nontrivial partition $\mathcal{E} \cup \mathcal{O}$. The next theorem gives a class of valid inequalities whose support graphs are antiweb-1-wheels. For $t = 2$ the simple



(a) (b)

FIGURE 5.3. (a) shows a simple $(8, 3)$-antiweb-1-wheel, where no edge is subdivided and all vertices (on the rim) belong to $\mathcal{O}$; (b) shows the nonsimple antiweb-1-wheel resulting from identifying vertices v_1 and v_5 in (a).

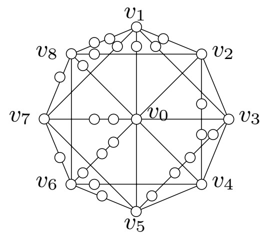


FIGURE 5.4. Picture of a simple $(8, 3)$-antiweb with $\mathcal{E} = \{v_1, v_6, v_8\}$ and $\mathcal{O} = \{v_2, v_3, v_4, v_5, v_7\}$.

(n, t)-antiweb-1-wheels and the inequalities given in the next theorem are just the 1-wheels and $\mathcal{I}_{\mathcal{E}}$ given in [CC97].

Theorem 5.5.2 (simple (n, t)-antiweb-1-wheel valid inequality).
Let $\mathcal{AWW}$ be an (n, t)-antiweb-1-wheel. Then the following inequality is valid for STAB$(\mathcal{AWW})$:

$$(I_{\mathcal{AWW}}) \qquad \left\lfloor \frac{n}{t} \right\rfloor x_0 + \sum_{i \in \mathcal{O}} x_i + (2t-2) \sum_{i \in \mathcal{E}} x_i + \sum_{i \in \mathcal{S} \cup \mathcal{R}} x_i \leq \left\lfloor \frac{n}{t} \right\rfloor + \frac{|\mathcal{S}| + |\mathcal{R}| + |\mathcal{E}|}{2} + (t-2)|\mathcal{E}|.$$

The class of $(\cdot, t)$-antiweb-1-wheels is referred to as (t)-antiweb-1-wheels and their corresponding inequalities are the (t)-antiweb-1-wheel inequalities. For the proof of validity, we need some intermediate results. Note that we purposely write K_{t+1} as $K_t^{v_0}$ in the next definition, and that $K_3^{v_0}$ is a 1-wheel in the sense of [CC97].

Definition 5.5.3 (odd subdivision of $(t \oplus 1)$-clique)**.**
Given $t > 2$, a t-clique K_t with vertex set $V = \{v_1, v_2, \ldots, v_t\}$ and a partition $\mathcal{E}, \mathcal{O}$ of V. Consider a subdivision G of $K_t^{v_0}$. Let $P_{0,i}$ denote the path obtained from subdividing the edge $\{v_0, v_i\}$ and let $P_{i,j}$ (for v_i, v_j adjacent in G_1) denote the path obtained from subdividing the edge $\{v_i, v_j\}$. This graph G is an odd subdivision of $(t \oplus 1)$-clique *if all of the following four conditions hold:*
1. *the length of $P_{0,i}$ is even for $i \in \mathcal{E}$ and odd for $i \in \mathcal{O}$,*
2. *the length of the path $P_{i,j}$ is even for $i \in \mathcal{E}$ and $j \in \mathcal{O}$ or $j \in \mathcal{E}$ and $i \in \mathcal{O}$,*
3. *the length of the path $P_{i,j}$ is odd for $i, j \in \mathcal{O}$, and*
4. *the length of the path $P_{i,j}$ is odd for $i, j \in \mathcal{E}$.*

Moreover, the internal vertices on the $P_{0,i}$'s give the set $\mathcal{S}$ and the internal vertices on the $P_{i,j}$'s give the set $\mathcal{R}$.

Lemma 5.5.4 ($(t \oplus 1)$-clique inequality)**.**
Given an odd subdivision of the $(t \oplus 1)$-clique $K_t^{v_0}$. Then the following inequality is valid

$$(I_{t\oplus 1}) \quad x_0 + (t-1)\sum_{i\in\mathcal{E}} x_i + \sum_{i\in\mathcal{O}} x_i + \sum_{v\in\mathcal{S}\cup\mathcal{R}} x_v \leq 1 + \frac{|\mathcal{S}| + |\mathcal{R}| + (t-2)|\mathcal{E}|}{2}.$$

PROOF. Notice, that $I_{t\oplus 1}$ reduces for $t = 2$ to an odd cycle inequality. So suppose now, that the validity of $I_{t'\oplus 1}$ is proved for all $t' < t$. Now consider G—an odd $K_{t\oplus 1}$. Notice that this graph contains t different $K_{(t-1)\oplus 1}$'s constructed by removing one spoke-end with all its incident paths. First we add up the $I_{(t-1)\oplus 1}$ inequalities for this subconfigurations and obtain:

$$tx_0 + (t-1)(t-2)\sum_{i\in\mathcal{E}} x_i + (t-1)\sum_{i\in\mathcal{O}} x_i + (t-1)\sum_{v\in\mathcal{S}} x_v + (t-2)\sum_{v\in\mathcal{R}} x_v \leq t + \frac{(t-1)|\mathcal{S}| + (t-2)|\mathcal{R}| + (t-1)(t-3)|\mathcal{E}|}{2}.$$

For each path P arising from an edge of the K_t, we construct $A(P)$ as follows (by looking at three cases): Let y and z be the spoke-ends of the path.

- Suppose $y, z \in \mathcal{E}$. If the path is $(y, u_1, u_2, \ldots, u_{2l}, z)$, where $l \geq 0$, we add the following edges to $A(P)$: $\{y, u_1\}, \{u_2, u_3\}, \ldots, \{u_{2l-2}, u_{2l-1}\}$, $\{u_{2l}, z\}$.
- Suppose $y, z \in \mathcal{O}$. If the path is $(y, u_1, u_2, \ldots, u_{2l}, z)$, where $l \geq 0$, we add the following edges to $A(P)$: $\{u_1, u_2\}, \{u_3, u_4\}, \ldots, \{u_{2l-1}, u_{2l}\}$.
- Suppose $y \in \mathcal{E}, z \in \mathcal{O}$. If the path is $(y, u_1, u_2, \ldots, u_{2l+1}, z)$, where $l \geq 0$, we add the edges $\{y, u_1\}, \{u_2, u_3\}, \ldots, \{u_{2l}, u_{2l+1}\}$ to $A(P)$.

Now we add for all $e \in A$ the edge inequalities and obtain

$$\begin{aligned} tx_0 &+ ((t-1)((t-2)+1)) \sum_{i \in \mathcal{E}} x_i \\ &+ (t-1) \sum_{i \in \mathcal{O}} x_i + (t-1) \sum_{v \in \mathcal{S}} x_v + (t-1) \sum_{v \in \mathcal{R}} x_v \\ &\leq t + \frac{(t-1)|\mathcal{S}| + (t-2)|\mathcal{R}| + (t-1)(t-3)|\mathcal{E}|}{2} + \frac{(t-1)|\mathcal{E}| + |\mathcal{R}|}{2}. \end{aligned}$$

Now divide everything by $t-1$

$$\begin{aligned} \frac{t}{t-1} x_0 + (t-1) \sum_{i \in \mathcal{E}} x_i + \sum_{i \in \mathcal{O}} x_i + \sum_{v \in \mathcal{S}} x_v + 1 \sum_{v \in \mathcal{R}} x_v \\ \leq \frac{t}{t-1} + \frac{|\mathcal{S}| + |\mathcal{R}| + (t-2)|\mathcal{E}|}{2}. \end{aligned}$$

Notice that $|\mathcal{S}| + |\mathcal{R}| + (t-2)|\mathcal{E}|$ is always even. Finally round first the coefficient of x_0 down and then the constant on the right hand side. This results in the inequality that needed to be proved. $\square$

For $t = 2$ the preceding inequality is a cycle inequality and for $t = 3$ it is a $\mathcal{I}_{\mathcal{E}}$, given in [CC97].

Lemma 5.5.5.
Given a simple antiweb-1-wheel. Then $|\mathcal{S}| + |\mathcal{R}| + |\mathcal{E}|$ is even.

PROOF. By the definition of $\mathcal{E}$, $|\mathcal{S}| + |\mathcal{E}|$ is even. Now consider a path $P_{i,j}$ corresponding to a cross-edge. If both ends are in $\mathcal{O}$, then the number of internal vertices of $P_{i,j}$ is even. If both ends are in $\mathcal{E}$, then the number of internal vertices of $P_{i,j}$ is even. If exactly one end of $P_{i,j}$ is in $\mathcal{E}$, then the number of internal vertices of $P_{i,j}$ is odd, and hence the number of internal vertices of $P_{i,j}$ + 1 is even. Hence $|\mathcal{R}| + b$ is even where b is the number of $P_{i,j}$'s with exactly one end in $\mathcal{E}$. Consider $|\mathcal{R}| + 2(t-1)|\mathcal{E}|$. Note that $2(t-1)$ is the number of $P_{i,j}$ an element of $\mathcal{E}$ is on. We now observe that $2(t-1)|\mathcal{E}| = 2a + b$ where a is the the number of $P_{i,j}$'s with both ends in $\mathcal{E}$. Hence b is even. This implies $|\mathcal{R}|$ is even. □

We are now ready to prove Theorem 5.5.2.

PROOF OF THEOREM 5.5.2. Given a simple antiweb-1-wheel, consider the subconfiguration (for a fixed i) generated by the paths $P_{0,i+1}$, $P_{0,i+2}, \ldots, P_{0,i+t}$ and $P_{i+j,i+k}$ for $1 \leq j, k, \leq t$ and $j \neq k$. This is an odd subdivision of $K_t^{v_0}$. We have n of these subconfigurations, one for each $i = 1, 2, \ldots, n$. Adding up the inequalities of the form in Lemma 5.5.4 for each of these subconfigurations gives

$$nx_0 + t(t-1)\sum_{v\in\mathcal{E}} x_v + t\sum_{v\in\mathcal{O}} x_v + t\sum_{v\in\mathcal{S}} x_v + \sum_{i=1}^{t-1}(t-i)\sum_{v\in\mathcal{R}_i} x_v \leq n + t\frac{|\mathcal{S}|}{2} + \sum_{i=1}^{t-1}(t-i)\frac{|\mathcal{R}_i|}{2} + \frac{t(t-2)}{2}|\mathcal{E}|$$

where $\mathcal{R}_i$ is the set of internal vertices of paths corresponding to the cross-edges of type i. For each path P arising from a cross-edge of type i define A as in the proof of Lemma 5.5.4 and add i times the edge inequality $x_u + x_v \leq 1$ for every $\{u, v\} \in A_i(P)$ thus obtaining

$$nx_0 + \left(t(t-1) + 2\sum_{i=1}^{t-1} i\right)\sum_{i\in\mathcal{E}} x_i + t\sum_{i\in\mathcal{O}} x_i + t\sum_{v\in\mathcal{S}} x_v + \sum_{i=1}^{t-1}(t-i+i)\sum_{v\in\mathcal{R}_i} x_v \leq n + t\frac{|\mathcal{S}|}{2} + \sum_{i=1}^{t-1}(t-i+i)\frac{|\mathcal{R}_i|}{2} + \frac{t(t-2)}{2}|\mathcal{E}| + 2\left(\sum_{i=1}^{t-1} i\right)\frac{|\mathcal{E}|}{2}.$$

This simplifies to

$$nx_0 + 2t(t-1)\sum_{i\in\mathcal{E}} x_i + t\sum_{i\in\mathcal{O}} x_i + t\sum_{v\in\mathcal{S}\cup\mathcal{R}} x_v \leq n + t\frac{|\mathcal{S}| + |\mathcal{R}| + |\mathcal{E}|}{2} + t(t-2)|\mathcal{E}|.$$

Dividing the inequality by t and rounding down (the term $|\mathcal{S}|+|\mathcal{R}|+|\mathcal{E}|$ is even by Lemma 5.5.5) gives the desired inequality,

$$\left\lfloor \frac{n}{t} \right\rfloor x_0 + 2(t-1)\sum_{i\in\mathcal{E}} x_i + \sum_{i\in\mathcal{O}} x_i + \sum_{v\in\mathcal{S}\cup\mathcal{R}} x_v \leq \left\lfloor \frac{n}{t} \right\rfloor + \frac{|\mathcal{S}|+|\mathcal{R}|+|\mathcal{E}|}{2} + (t-2)|\mathcal{E}|. \qquad \square$$

We note that we can extend all these inequalities to include the inequalities for the corresponding nonsimple configurations by Lemma 5.2.1. As before, such inclusion is crucial for our separation algorithms. Of course for a nonsimple antiweb-1-wheel inequality, sets such as $\mathcal{S}$, $\mathcal{R}$ and $\mathcal{E}$ are multisets using the same rule as before.

5.6. Separation Algorithms

Let $t \geq 2$. We separate (t)-antiweb-1-wheel inequalities. (The case $t = 2$ is given in [CC97].) We need to rewrite the inequality in Lemma 5.5.4 in terms of the w_e's. We use the notation

$$f(v) = \begin{cases} 1/4 - x_v/2 & \text{if } v \in \mathcal{E} \\ -1/4 + x_v/2 & \text{if } v \in \mathcal{O} \end{cases}$$

for $v \in \mathcal{E} \cup \mathcal{O}$. This f is an extension of the definition of f we gave earlier (where for antiwebs every vertex corresponds to a vertex in $\mathcal{O}$ as its coefficient is 1). Again, we use f^* to denote f when x is replaced by x^*. From now on, we assume x^* satisfies the trivial, edge and cycle inequalities.

Lemma 5.6.1.
For every path $P_{a,b}$ arising from an i-edge of an antiweb holds

$$2\sum_{e\in A_i(P_{a,b})} w_e = \sum_{e\in P_{a,b}} w_e + f(v_a) + f(v_b).$$

PROOF. This follows from the definitions of w_e and $A_i(P_{a,b})$. $\square$

Using $w(P) = \sum_{e\in P} w_e$ we obtain the following path-representation of $I_{t\oplus 1}$.

Lemma 5.6.2.
A $(t \oplus 1)$-clique inequality can be rewritten as

$$I_{t\oplus 1} : -\sum_{P\in\mathcal{P}(S)} w(P) - \sum_{P\in\mathcal{P}(R)} w(P) - (t-2)\sum_{v\in\mathcal{E}\cup\mathcal{O}} f(v) + \left(\frac{3t}{4} - 1\right) - \frac{t-2}{2}x_0 \leq 0$$

where $\mathcal{P}(S)$ is the set of paths obtained by replacing the edges of the form (v_0, v) where $v \in V(K_t)$, and $\mathcal{P}(R)$ is the set of paths obtained by replacing the edges of the form (u, v) where $u, v \in V(K_t)$. Moreover, for fixed t the $(t \oplus 1)$-clique inequalities can be separated with respect to $CSTAB(G)$ in polynomial time.

Again, the term $(t\oplus 1)$-clique inequality in Lemma 5.6.2 refers to simple and nonsimple instances. As before, the inequality should be interpreted in the usual way for nonsimple configurations: The number of times w_e appears is according to the number of roles it takes as multiple edges are deleted.

PROOF OF LEMMA 5.6.2. The part of the theorem about the representation is simply done by comparing the corresponding terms in the two representations. Consider the multivariable function

$$\phi = -\sum_{P\in\mathcal{P}(S)} w(P) - \sum_{P\in\mathcal{P}(R)} w(P) - (t-2)\sum_{v\in\mathcal{E}\cup\mathcal{O}} f(v) + \left(\frac{3t}{4} - 1\right) - \frac{t-2}{2}x_0$$

and compare the coefficients of the terms.

1. $[x_0]\phi$: As only the first and last coefficient contribute, we obtain $t/2 - (t-2)/2 = 1$.
2. $[x_v]\phi$ *where v is an internal vertex in $P \in \mathcal{P}(S) \cup \mathcal{P}(R)$:* As the degree of x_v is two and x_v appears in either $-\sum_{P\in\mathcal{P}(S)} w(P)$ or $-\sum_{P\in\mathcal{P}(R)} w(P)$ only, $[x_v]\phi = 1/2 + 1/2 = 1$.
3. $[x_v]\phi$ *where $v \in \mathcal{E}$:* Then we obtain $[x_v](-\sum_{P\in\mathcal{P}(S)} w(P)) = 1/2$, $[x_v](\sum_{P\in\mathcal{P}(R)} w(P)) = (t-1)/2$ and $[x_v]f(v) = -1/2$. Hence $[x_v]\phi = 1/2 + (t-1)/2 + (t-2)/2 = t-1$.
4. $[x_v]\phi$ *where $v \in \mathcal{O}$:* The terms are the same as in the last case just the last sign is opposite so that we obtain $[x_v]\phi = \frac{1}{2} + \frac{t-1}{2} - \frac{t-2}{2} = 1$.

5. [1]ϕ: Notice that the first term of ϕ contributes $-(t+|\mathcal{S}|)/2$ the second term $-(2|\mathcal{R}|+t(t-1))/4$ the third term $-(|\mathcal{E}|-|\mathcal{O}|)*(t-2)/4$ and the fourth term contributes $-1+3t/4$. Adding them up and using $|\mathcal{O}|=t-|\mathcal{E}|$ yields the desired $-1-(|\mathcal{S}|+|\mathcal{R}|+(t-2)|\mathcal{E}|)/2$.

The first part implies the following: If $K_t^{v_0}$ determines such an inequality that is most-violated by x^*, then every path in $\mathcal{P}(S)\cup\mathcal{P}(R)$ is a minimum-weight walk with respect to w^* of its parity joining its ends. We need a method of separating these inequalities (both simple and nonsimple) for a fixed t. We compute, for each $u,v\in V$, the minimum weight with respect to w^* of an even (odd) walk from u to v in G (u and v may be the same). We denote this minimum by $w_E^*(u,v)$ ($w_O^*(u,v)$). To solve our problem for $I_{t\oplus 1}$, it is enough to find an algorithm for finding a most-violated inequality of the form $I_{t\oplus 1}$ with some specific vertex, say v_0, as the hub. We construct an auxiliary graph $H=(V_H,E_H)$ from $G=(V,E)$ as follows: H is a complete graph with loops where $V_H=V^{\mathcal{E}}\cup V^{\mathcal{O}}$, and $V^{\mathcal{E}}$ and $V^{\mathcal{O}}$ are copies of V. If $a\in V^{\mathcal{E}}$ ($V^{\mathcal{O}}$) is a copy of $b\in V$, then b is denoted by α_a. A vertex in $V^{\mathcal{E}}$ represents a potential even spoke-end and a vertex in $V^{\mathcal{O}}$ represents a potential odd spoke-end. We first define the following: Given $u,v\in V_H$, we set

$$w^+(u,v)=\begin{cases} w_O^*(\alpha_u,\alpha_v) & \text{if } u,v \text{ belong to the same set of } V^{\mathcal{E}} \text{ and } V^{\mathcal{O}},\\ w_E^*(\alpha_u,\alpha_v) & \text{if } u,v \text{ belong to different sets of } V^{\mathcal{E}} \text{ and } V^{\mathcal{O}};\end{cases}$$

and $f^+(u)=f^*(\alpha_u)$. Additionally, we need a way to describe the distance of an even or odd path from the hub v_0 to another vertex u. Of course, $w_E^*(v_0,u)$ and $w_O^*(v_0,u)$ describe this in G. A neat way to describe it in H is to notice that $w^+(v_0^{\mathcal{O}},u)$ is equal to $w_E^*(v_0,\alpha_u)$ if $u\in V^{\mathcal{E}}$ (as $v_0^{\mathcal{O}}$ and u belong to different sets of $\mathcal{E}$ and $\mathcal{O}$) and $w_O^*(v_0,\alpha_u)$ if $u\in V^{\mathcal{O}}$.

We let the weight on the edge $\{u,v\}$ be

$$\begin{aligned} w^+(u,v)+\frac{1}{t-1}w^+(v_0^{\mathcal{O}},u)+\frac{1}{t-1}w^+(v_0^{\mathcal{O}},v)\\ +\frac{t-2}{t-1}(f^+(u)+f^+(v))-\frac{1}{2(t-1)}.\end{aligned}$$

This is nonnegative because it can be rewritten as

$$\frac{1}{t-1}\left(w^+(u,v)+w^+(v_0^{\mathcal{O}},u)+w^+(v_0^{\mathcal{O}},v)-\frac{1}{2}\right) \\ +\frac{t-2}{t-1}(w^+(u,v)+f^+(u)+f^+(v)).$$

For the first term to be nonnegative it suffices to show that x^* fulfills the corresponding odd cycle inequality, and this is fulfilled by assumption. The second line is nonnegative by Lemma 5.6.1. Now consider a (possibly nonsimple) K_t in H. (We note that a K_{t-1} with a loop at a vertex is a nonsimple K_t.) The weight of this K_t is

$$\sum_{P\in\mathcal{P}(S)} w^+(P)+\sum_{P\in\mathcal{P}(R)} w^+(P)+(t-2)\sum_{v\in\mathcal{E}\cup\mathcal{O}} f^+(v)-\frac{t}{4}.$$

So we find the weight of a minimum-weight nonsimple K_t and there is a violated odd-clique inequality if and only if its value is less than $\frac{t-2}{2}(1-x_0^*)$ by the first part. As t is fixed, it suffices to inspect all $O(|V|^t)$ of these t-tuples to decide whether x^* violates any of the odd $(t\oplus 1)$-clique inequalities for a single, fixed hub; this gives a polynomial time separation algorithm. ☐

Lemma 5.6.2 implies that we may from now on assume that x^* fulfills all $(t\oplus 1)$-clique inequalities. We denote with $\mathrm{Q}'_t\mathrm{STAB}(G)$ the set of all $x\in \mathrm{ESTAB}(G)$ that fulfill all $(t\oplus 1)$-clique inequalities. We are now ready to approach the antiweb-1-wheel separation itself. We first rewrite the inequality of Theorem 5.5.2 into its path-representation.

Lemma 5.6.3.
An antiweb-1-wheel inequality $I_{\mathcal{AWW}}$ can be written as

$$-\sum_{P\in\mathcal{P}(S)} w(P)-\sum_{i=1}^{t-1}\sum_{P\in\mathcal{P}(J_i)} w(P)-(2t-3)\sum_{v\in\mathcal{E}\cup\mathcal{O}} f(v) \\ -\left(\frac{n}{2}-\left\lfloor\frac{n}{t}\right\rfloor\right)x_0+\left(\frac{3}{4}n-\left\lfloor\frac{n}{t}\right\rfloor\right)\le 0$$

where $\mathcal{P}(S)$ and $\mathcal{P}(J_i)$ are the sets of paths derived from the spokes and i-edges, respectively.

Proof. It is clear that we only have to prove this result for the case where the antiweb-1-wheel is simple, the nonsimple case follows directly.

Consider the multivariable function

$$\phi = -\sum_{P\in\mathcal{P}(S)} w(P) - \sum_{i=1}^{t-1}\sum_{P\in\mathcal{P}(J_i)} w(P) - (2t-3)\sum_{v\in\mathcal{E}\cup\mathcal{O}} f(v) \\ - \left(\frac{n}{2} - \left\lfloor\frac{n}{t}\right\rfloor\right) x_0 + \left(\frac{3}{4}n - \left\lfloor\frac{n}{t}\right\rfloor\right)$$

and compare the coefficients of the terms.

1. $[x_0]\phi$: We have $[x_0]\phi = [x_0](-\sum_{P\in\mathcal{P}(S)} w(P)) - (n/2 - \lfloor n/t\rfloor) = n/2 - n/2 + \lfloor n/t\rfloor = \lfloor n/t\rfloor$.
2. $[x_v]\phi$ *where* v *is an internal vertex in* $P \in \mathcal{P}(S)\cup\mathcal{P}(J_1)\cup\cdots\cup\mathcal{P}(J_t)$*:* Because the degree of x_v is two and x_v appears in either $-\sum_{P\in\mathcal{P}(S)} w(P)$ or $-\sum_{i=1}^{t-1}\sum_{P\in\mathcal{P}(J_i)} w(P)$ only, $[x_v]\phi = 1/2 + 1/2 = 1$.
3. $[x_v]\phi$ *where* $v\in\mathcal{E}$*:* Then we obtain $[x_v](-\sum_{P\in\mathcal{P}(S)} w(P)) = 1/2$, $[x_v](-\sum_{i=1}^{t-1}\sum_{P\in\mathcal{P}(J_i)} w(P)) = (2t-2)/2$ and $[x_v]f(v) = -1/2$. Hence $[x_v]\phi = 1/2 + (2t-2)/2 + (2t-3)/2 = 2t-2$.
4. $[x_v]\phi$ *where* $v\in\mathcal{O}$*:* Then we obtain $[x_v](-\sum_{P\in\mathcal{P}(S)} w(P)) = 1/2$, $[x_v](-\sum_{i=1}^{t-1}\sum_{P\in\mathcal{P}(J_i)} w(P)) = (2t-2)/2$ and $[x_v]f(v) = 1/2$. Hence $[x_v]\phi = 1/2 + (2t-2)/2 - (2t-3)/2 = 1$.
5. $[1]\phi$*:* We note that $[1](\sum_{P\in\mathcal{P}(S)} w(P) + \sum_{i=1}^{t-1}\sum_{P\in\mathcal{P}(J_i)} w(P))$ is equal to $1/2$ times the number of edges in the antiweb-1-wheel. The number of edges in the antiweb-1-wheel is $n + n(t-1) + |\mathcal{S}| + |\mathcal{R}| = nt + |\mathcal{S}| + |\mathcal{R}|$. Moreover, we observe that $[1]f(v) = 1/4$ if $v\in\mathcal{E}$ and $[1]f(v) = -1/4$ if $v\in\mathcal{O}$. Hence

$$[1]\phi = -\frac{nt+|\mathcal{S}|+|\mathcal{R}|}{2} - \frac{2t-3}{4}|\mathcal{E}| + \frac{2t-3}{4}|\mathcal{O}| + \frac{3}{4}n - \left\lfloor\frac{n}{t}\right\rfloor \\ = -\left\lfloor\frac{n}{t}\right\rfloor - \frac{|\mathcal{S}|+|\mathcal{R}|+|\mathcal{E}|}{2} - (t-2)|\mathcal{E}|$$

as $|\mathcal{O}| = n - |\mathcal{E}|$. □

Corollary 5.6.4.
Let W determine an inequality of the form $I_{\mathcal{AWW}}$ that is most-violated by x^*. Then every path in $\mathcal{P}(S)\cup\mathcal{P}(J_1)\cup\mathcal{P}(J_2)\cup\cdots\cup\mathcal{P}(J_{t-1})$ is a minimum-weight walk with respect to w^* of its parity joining its ends. In other words, if P is such a (nonempty) walk, and P joins a to b, then for every nonempty

walk Q from a to b having the same parity as P, the weight of Q is at least the weight of P with respect to w^*.

PROOF. With

$$F = \sum_{P \in \mathcal{P}(S)} w(P) + \sum_{i=1}^{t-1} \sum_{P \in \mathcal{P}(J_i)} w(P) + (2t-3) \sum_{v \in \mathcal{E} \cup \mathcal{O}} f(v) + \left(\frac{n}{2} - \frac{n}{t}\right) x_0 - \left(\frac{3}{4}n - \frac{n}{t}\right)$$

we rewrite $I_{\mathcal{AWW}}$ as $F \geq \left(\frac{n}{t} - \left\lfloor \frac{n}{t} \right\rfloor\right)(1 - x_0)$ (from Lemma 5.6.3). □

Motivated by Corollary 5.6.4, we compute, for each $u, v \in V$, the minimum weight with respect to w^* of an even (odd) nonempty walk from u to v in G (u and v may be the same). We denote this minimum by $w_E^*(u,v)$ and $w_O^*(u,v)$. To solve the separation problem for $I_{\mathcal{AWW}}$, it is enough to find an algorithm for finding a most-violated inequality of the form $I_{\mathcal{AWW}}$ with some specific vertex, say v_0, as the hub. We construct an auxiliary graph $H = (V_H, E_H)$ from $G = (V, E)$ as follows: H is a complete graph with loops where $V_H = V^{\mathcal{E}} \cup V^{\mathcal{O}}$, and $V^{\mathcal{E}}$ and $V^{\mathcal{O}}$ are copies of V. If $a \in V^{\mathcal{E}}$ ($V^{\mathcal{O}}$) is a copy of b, then b is denoted by α_a. A vertex in $V^{\mathcal{E}}$ represents a potential even spoke-end and a vertex in $V^{\mathcal{O}}$ represents a potential odd spoke-end. Here, we will assign weights to ordered t-cliques rather than edges. We first define the following: Given $u, v \in V_H$, we set

$$w^+(u,v) = \begin{cases} w_O^*(\alpha_u, \alpha_v) & \text{if } u, v \text{ belong to the same set of } V^{\mathcal{E}} \text{ and } V^{\mathcal{O}}, \\ w_E^*(\alpha_u, \alpha_v) & \text{if } u, v \text{ belong to different sets of } V^{\mathcal{E}} \text{ and } V^{\mathcal{O}}; \end{cases}$$

for $u \in V_H$, we set $w^+(v_0, u) = w_E^*(v_0, \alpha_u)$ if $u \in V^{\mathcal{E}}$ and $w^+(v_0, u) = w_O^*(v_0, \alpha_u)$ if $u \in V^{\mathcal{O}}$. Moreover, we let $f^+(u) = f^*(\alpha_u)$. For the ordered t-clique (possibly degenerate) $(u_1, u_2, \ldots, u_t)$, we assign the following weight to it:

$$\frac{1}{t} \sum_{i=1}^{t} w^+(v_0, u_i) + \sum_{1 \leq j < i \leq t} \left(\frac{1}{t-(i-j)}\right) w^+(u_i, u_j) - \frac{3}{4} + \frac{1}{t} + \frac{t-2}{2t} x_0^* + \frac{t-2}{t} \sum_{i=1}^{t} f^+(u_i) + \sum_{1 \leq j < i \leq t} \left(\frac{1}{t-(i-j)} - \frac{1}{t}\right) \left(f^+(u_i) + f^+(u_j)\right).$$

We claim that this weight is nonnegative. We rewrite it as

$$\frac{1}{t}\sum_{i=1}^{t} w^{+}(v_0,u_i)+\frac{1}{t}\sum_{1\le j<i\le t} w^{+}(u_i,u_j)-\frac{3}{4}+\frac{1}{t}+\frac{t-2}{2t}x_0+\frac{t-2}{t}\sum_{i=1}^{t} f(u_i)$$
$$+\sum_{1\le j<i\le t}\left(\frac{1}{t-(i-j)}-\frac{1}{t}\right)(w^{*}(u_i,u_j)+f(u_i)+f(u_j)).$$

The first line is nonnegative as it is proportional to the form given in Lemma 5.6.2 and as $x^* \in \mathrm{Q}'_t\mathrm{STAB}(G)$ we know that x^* satisfies $(t \oplus 1)$-clique inequalities. Each summand in the second line is nonnegative by Lemma 5.6.1.

Suppose we have n ordered t-cliques, $(u_1^1, u_2^1, \ldots, u_t^1), (u_1^2, u_2^2, \ldots, u_t^2), \ldots, (u_1^n, u_2^n, \ldots, u_t^n)$, such that $u_{j+1}^i = u_j^{i+1}$ and $u_1^i \neq u_t^{i+1}$ for $i = 1, 2, \ldots, n-1$ and $j = 1, 2, \ldots, t-1$, and $u_j^n = u_{j+1}^1$ for $j = 1, 2, \ldots, t-1$ and $u_t^n \neq u_1^1$. We note that these t-cliques generate a possibly degenerate (n,t)-antiweb. Unlike the proof of Theorem 5.3.2, a degenerate (n,t)-antiweb poses no problem as a loop induces a nonempty walk in the original graph. However, they may induce a nonsimple (n,t)-antiweb-1-wheel in the original graph. The total weight of these n ordered t-cliques is

$$\sum_{P\in\mathcal{P}(S)} w^{+}(P)+\sum_{i=1}^{t-1}\sum_{P\in\mathcal{P}(J_i)} w^{+}(P)+(2t-3)\sum_{v\in\mathcal{E}\cup\mathcal{O}} f^{+}(v)$$
$$+\left(\frac{n}{2}-\frac{n}{t}\right)x_0^{+}-\left(\frac{3}{4}n-\frac{n}{t}\right).$$

To see this, we first observe that we only have to prove the claim in which the resulting (n,t)-antiweb is simple; the nonsimple case follows immediately. We proceed as follow.

1. For a given u_i, $w^{+}(v_0,u_i)$ appears with weight $\frac{1}{t}$ in each of the exactly t of the n t-tuples; this gives $w^{+}(v_0,u_i)$ for the total contribution.
2. For a given u_i and u_j, with $j < i$, where (u_i,u_j) is a k-edge in the generated (n,t)-antiweb. Then it appears in $t-k$ of the t-tuples; this gives $w^{+}(u_i,u_j)$ for the total contribution as $k = i-j$.
3. Let $u \in \mathcal{E}\cup\mathcal{O}$. Then u appears in exactly t of the n t-tuples. Moreover, it appears as the ith entry of a t-tuple exactly once for every $1 \le i \le t$.

Hence the total contribution is

$$\left(\frac{t-2}{t}\sum_{i=1}^{t} 1 + 2\sum_{1\leq j<i\leq t}\left(\frac{1}{t-(i-j)} - \frac{1}{t}\right)\right) f^+(u) = (2t-3)f^+(u).$$

4. As x_0^* appears with weight $t-2/2t$ in each of the n t-tuples, this gives $(n/2 - n/t)x_0^*$ for the total contribution.
5. The constant term in each of the n t-tuples is $-3/4 + 1/t$ hence the total contribution is $-3n/4 + n/t$

Hence the claim is established.

This gives a violated inequality for the corresponding antiweb-1-wheel if and only if this value is less than $\left(\frac{n}{t} - \left\lfloor\frac{n}{t}\right\rfloor\right)(1-x_0^*)$. So we have proved the next theorem.

Lemma 5.6.5.
If A is a minimum-weight (t)-antiweb in H of size congruent to $q \bmod t$, then x^* satisfies all (n,t)-antiweb-1-wheel inequalities with n congruent to q modulo t and v_0 as the hub if and only if the weight of A is at least $\left(\frac{n}{t} - \left\lfloor\frac{n}{t}\right\rfloor\right)(1-x_0^*)$.

Therefore, we want to find such a minimum weight (t)-antiweb (possibly degenerate) generated by these t-cliques in H. The next corollary is obtained by applying Theorem 5.3.1 for each $q = 1, 2, \ldots, t-1$ and every possible hub to Lemma 5.6.5.

Theorem 5.6.6.
For fixed t holds that the separation problem for (t)-antiweb-1-wheel inequalities with respect to $Q_t'STAB(G)$ can be solved in polynomial time.

5.7. Antiweb-s-Wheel Inequalities and their Separation

In this section we generalize the notion of antiweb-1-wheel by permitting larger cliques for the hub.

Definition 5.7.1 (simple antiweb-s-wheel)**.**
Given an (n,t)-antiweb $G_1 = (V_1, E_1)$ with vertex set $V_1 = \{v_1, v_2, \ldots, v_n\}$, a partition of V_1 into $\mathcal{E}$ and $\mathcal{O}$, and $s \geq 0$. Consider a subdivision G of $G_1^{v_{0_1}, v_{0_2}, \ldots, v_{0_s}}$. Let $P_{0_i,j}$ denote the path obtained from subdividing the edge $\{v_{0_i}, v_j\}$ (called a spoke*), and let $P_{i,j}$ (for v_i, v_j adjacent in G_1) denote the*

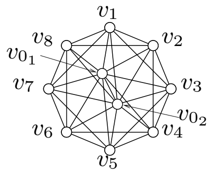


FIGURE 5.5. Picture of a simple $(8,3)$-antiweb-2-wheel, where no edge is subdivided and all vertices (on the rim) belong to $\mathcal{O}$.

path obtained from subdividing the edge $\{v_i, v_j\}$. *The resulting graph* $\mathcal{AWW}$ *is a* simple antiweb-s-wheel *if it satisfies the following conditions:*

1. *For all* $i \in \{1, 2, \dots, s\}$ *is the length of* $P_{0_i,j}$ *even for* $j \in \mathcal{E}$ *and odd for* $j \in \mathcal{O}$;
2. *the length of the path* $P_{i,j}$ *is even for* $i \in \mathcal{E}$ *and* $j \in \mathcal{O}$ *or* $j \in \mathcal{E}$ *and* $i \in \mathcal{O}$;
3. *the length of the path* $P_{i,j}$ *is odd for* $i, j \in \mathcal{O}$; *and*
4. *the length of the path* $P_{i,j}$ *is odd for* $i, j \in \mathcal{E}$.

(Note, that edges between vertices of the hub-set $\{v_{0_1}, v_{0_2}, \dots, v_{0_s}\}$ *cannot be subdivided.) Moreover let* $\mathcal{S}(\mathcal{AWW})$, *or simply* $\mathcal{S}$, *be the set of internal vertices of* $P_{0_i,j}$ *for* $i = 1, 2, \dots, s$, $j = 1, 2, \dots, n$, *and* $\mathcal{R}(\mathcal{AWW})$, *or simply* $\mathcal{R}$, *be the set of internal vertices of all the* $P_{i,j}$*'s for all* $i, j \in \{1, 2, \dots, n\}$. *The set of vertices* $\{v_{0_1}, v_{0_2}, \dots, v_{0_s}\}$ *constitutes the* hubset *of the antiweb-s-wheel, and the vertices in* $\mathcal{E} \cup \mathcal{O}$ *are the* spoke-ends.

See Figure 5.7 for a simple antiweb-2-wheel. If $t = 2$, the antiweb-s-wheel is just a s-wheel defined in [CC97]. For the proof of validity of their underlying inequalities we need again to study a subdivision of cliques. This time we purposely write K_{t+s} as $K_t^{v_{0_1}, v_{0_2}, \dots, v_{0_s}}$ in the next definition.

Definition 5.7.2 (odd subdivision of $(t \oplus s)$-clique)**.**
Given a $(t+s)$*-clique* $K_t^{v_{0_1}, v_{0_2}, \dots, v_{0_s}}$ *with vertex-set* $\{v_{0_1}, v_{0_2}, \dots, v_{0_s}\} \cup V$ *where* $V = \{v_1, v_2, \dots, v_t\}$ *and a partition* $\mathcal{E}, \mathcal{O}$ *of* V. *Consider a subdivision* G *of* $K_t^{v_{0_1}, v_{0_2}, \dots, v_{0_s}}$. *Let* $P_{0_i,j}$ *denote the path obtained from subdividing the edge* $\{v_{0_i}, v_j\}$ *and let* $P_{i,j}$ *(for* v_i, v_j *adjacent in* G_1*) denote the path obtained from subdividing the edge* $\{v_i, v_j\}$. *This graph* G *is an* odd subdivision of $(t \oplus s)$-clique *if all of the following four conditions hold:*

1. *for all* $i \in \{1, 2, \ldots, s\}$ *is the length of* $P_{0_i,j}$ *even for* $j \in \mathcal{E}$ *and odd for* $j \in \mathcal{O}$,
2. *the length of the path* $P_{i,j}$ *is even for* $i \in \mathcal{E}$ *and* $j \in \mathcal{O}$ *or* $j \in \mathcal{E}$ *and* $i \in \mathcal{O}$,
3. *the length of the path* $P_{i,j}$ *is odd for* $i, j \in \mathcal{O}$, *and*
4. *the length of the path* $P_{i,j}$ *is odd for* $i, j \in \mathcal{E}$.

(Note, that edges between vertices of the hub-set $\{v_{0_1}, v_{0_2}, \ldots, v_{0_s}\}$ *cannot be subdivided.) Moreover, the internal vertices on the* $P_{0_i,j}$*'s give the set* $\mathcal{S}$ *and the internal vertices on the* $P_{i,j}$*'s give the set* $\mathcal{R}$.

Lemma 5.7.3 (($t \oplus s$)-clique inequality)**.**
Given an odd subdivision of the ($t \oplus s$)*-clique* $K_t^{v_{0_1}, v_{0_2}, \ldots, v_{0_s}}$. *Then the following inequality is valid*

$$(I_{t\oplus s}) \quad \sum_{i=1}^{s} x_{0_i} + (t-2+s)\sum_{i\in\mathcal{E}} x_i + \sum_{i\in\mathcal{O}} x_i + \sum_{v\in\mathcal{S}\cup\mathcal{R}} x_v \leq 1 + \frac{|\mathcal{S}| + |\mathcal{R}| + (t-3+s)|\mathcal{E}|}{2}.$$

All $x \in Q'_{(t+s-1)\oplus 1}STAB(G)$ *fulfill all* ($t \oplus s$)*-clique inequalities.*

PROOF. For the proof it is only important to realize, that every ($t \oplus s$)-clique is also a (($t + s - 1) \oplus 1$)-clique. Consider a ($t \oplus s$)-clique G with given vertex partition $\mathcal{E}\dot{\cup}\mathcal{O}$. Now notice that G is at the same time a (($t + s - 1) \oplus 1$)-clique with hub v_{0_1} and partition $\mathcal{E}' = \mathcal{E}$ and $\mathcal{O}' = \mathcal{O} \cup \{v_{0_2}, v_{0_3}, \ldots, v_{0_s}\}$ for the following reasons:

1. the paths between v_{0_1} and $\mathcal{E}$ and $\mathcal{O}$ behave properly as they did so in the ($t \oplus s$)-clique;
2. the paths between v_{0_1} and $\{v_{0_2}, v_{0_3}, \ldots, v_{0_s}\}$ are of length 1 and therefore odd;
3. the paths from $\{v_{0_2}, v_{0_3}, \ldots, v_{0_s}\}$ to $\mathcal{O}$ are odd as they are in the (($t + s - 1) \oplus 1$)-clique;
4. the paths from $\{v_{0_2}, v_{0_3}, \ldots, v_{0_s}\}$ to $\mathcal{E}$ are even as they are in the (($t + s - 1) \oplus 1$)-clique.

So we have established that G is also a (($t + s - 1) \oplus 1$)-clique. Now it suffices to observe that $I_{t\oplus s}$ is just the same as $I_{(t+s-1)\oplus 1}$ so that validity follows by Lemma 5.5.4. The second claim is now trivial. □

The second part of Lemma 5.7.3 implies that after separating all (($t + s - 1) \oplus 1$)-clique inequalities no ($t \oplus s$)-clique inequality could be violated.

Hence for fixed s, t we can from now on assume that for x^* all $(t \oplus s)$-clique inequalities are fulfilled.

Next we need a path representation for the $(t \oplus s)$-clique inequalities. Unfortunately, this takes more effort than the preceding proof, as the representation cannot be derived directly from the result about $(t \oplus 1)$-cliques.

Lemma 5.7.4.
A $(t \oplus s)$-clique inequality can be rewritten as

$$I_{t\oplus s} : -\sum_{P\in\mathcal{P}(S)} w(P) - \sum_{P\in\mathcal{P}(R)} w(P) - (t-3+s)\sum_{v\in\mathcal{E}\cup\mathcal{O}} f(v) + \left(\frac{t(s+2)}{4} - 1\right) - \frac{t-2}{2}\sum_{i=1}^{s} x_{0_i} \le 0$$

where $\mathcal{P}(S)$ are the set of paths obtained by replacing the edges of the form (v_0, v) where $v \in V(K_t)$, and $\mathcal{P}(R)$ the set of paths obtained by replacing the edges of the form (u, v) where $u, v \in V(K_t)$.

Again, the term $(t \oplus s)$-clique inequality in Lemma 5.7.4 refers to simple and nonsimple instances. As before, the inequality should be interpreted in the usual way for nonsimple configurations: The number of times w_e appears is according to the number of roles it takes as multiple edges are deleted.

PROOF OF LEMMA 5.7.4. The part of the theorem about the representation is simply done by comparing the corresponding terms in the two representations. Consider the multivariable function

$$\phi = -\sum_{P\in\mathcal{P}(S)} w(P) - \sum_{P\in\mathcal{P}(R)} w(P) - (t-3+s)\sum_{v\in\mathcal{E}\cup\mathcal{O}} f(v) + \left(\frac{t(s+2)}{4} - 1\right) - \frac{t-2}{2}\sum_{i=1}^{s} x_{0_i}$$

and compare the coefficients of the terms.

1. $[x_{0_i}]\phi$*:* As only the first and last coefficient contribute, we obtain $t/2 - (t-2)/2 = 1$.
2. $[x_v]\phi$ *where v is an internal vertex in $P \in \mathcal{P}(S) \cup \mathcal{P}(R)$:* As the degree of x_v is two and x_v appears in either $-\sum_{P\in\mathcal{P}(S)} w(P)$ or $-\sum_{P\in\mathcal{P}(R)} w(P)$ only, $[x_v]\phi = 1/2 + 1/2 = 1$.

3. $[x_v]\phi$ *where* $v \in \mathcal{E}$*:* Then we obtain $[x_v](-\sum_{P\in\mathcal{P}(S)} w(P)) = s/2$, $[x_v](\sum_{P\in\mathcal{P}(R)} w(P)) = (t-1)/2$ and $[x_v]f(v) = -1/2$. Hence $[x_v]\phi = s/2 + (t-1)/2 + (t-3+s)/2 = (t-2+s)$.
4. $[x_v]\phi$ *where* $v \in \mathcal{O}$*:* The terms are the same as in the last case just the last term has opposite sign such that we obtain $[x_v]\phi = s/2 + (t-1)/2 - (t-3+s)/2 = 1$.
5. $[1]\phi$*:* Notice that the first term of ϕ contributes $-(st + |\mathcal{S}|)/2$ the second term $-(2|\mathcal{R}|+t(t-1))/4$ the third term $-(|\mathcal{E}|-|\mathcal{O}|)*(t-3+s)/4$ and the fourth term contributes $-1+t(2+s)/4$. Adding them up and using $|\mathcal{O}| = t - |\mathcal{E}|$ yields the desired $-1-(|\mathcal{S}|+|\mathcal{R}|+(t-3+s)|\mathcal{E}|)/2$. □

The next theorem gives a class of valid inequalities whose support graphs are antiweb-s-wheels. For $t = 2$, this class reduces to $\mathcal{I}_\mathcal{E}$ given in [CC97].

Theorem 5.7.5 (simple (n,t)-antiweb-s-wheel valid inequality)**.**
Let $\mathcal{AWW}$ be an (n,t)-antiweb-s-wheel. Then the following inequality is valid for STAB($\mathcal{AWW}$):

$$(I_{\mathcal{AWW}}) \qquad \left\lfloor \frac{n}{t} \right\rfloor \sum_{i=1}^{s} x_{0_i} + \sum_{i\in\mathcal{O}} x_i + (2t-3+s)\sum_{i\in\mathcal{E}} x_i + \sum_{v\in\mathcal{S}\cup\mathcal{R}} x_v \leq \left\lfloor \frac{n}{t} \right\rfloor + \frac{|\mathcal{S}|+|\mathcal{R}|+(2t-4+s)|\mathcal{E}|}{2}$$

The class of $(\cdot,t)$-antiweb-s-wheels is referred to as (t)-antiweb-s-wheels and their corresponding inequalities are the (t)-antiweb-s-wheels inequalities. For the proof of validity, we need an additional lemma.

Lemma 5.7.6.
Given a simple antiweb-s-wheel. Then $|\mathcal{S}|+|\mathcal{R}|+(2t-4+s)|\mathcal{E}|$ is even.

PROOF. By the definition of $\mathcal{E}$, the term $|\mathcal{S}|+s|\mathcal{E}|$ is even. Obviously $(2t-4)|\mathcal{E}|$ is even. Now consider a path $P_{i,j}$ corresponding to a cross-edge. If both ends are in $\mathcal{O}$, then the number of internal vertices of $P_{i,j}$ is even. If both ends are in $\mathcal{E}$, then the number of internal vertices of $P_{i,j}$ is even. If exactly one end of $P_{i,j}$ is in $\mathcal{E}$, then the number of internal vertices of $P_{i,j}$ is odd, and hence the number of internal vertices of $P_{i,j}$ + 1 is even. Hence $|\mathcal{R}|+b$ is even where b is the number of $P_{i,j}$'s with exactly one end in $\mathcal{E}$. Consider $|\mathcal{R}|+2(t-1)|\mathcal{E}|$. Note that $2(t-1)$ is the number of $P_{i,j}$

an element of $\mathcal{E}$ is on. We now observe that $2(t-1)|\mathcal{E}| = 2a + b$ where a is the the number of $P_{i,j}$'s with both ends in $\mathcal{E}$. Hence b is even. This implies $|\mathcal{R}|$ is even. $\square$

We are now ready to prove Theorem 5.7.5.

PROOF OF THEOREM 5.7.5. Given a simple antiweb-s-wheel, consider the subconfiguration (for a fixed i) generated by the paths $P_{0_l,i+1}$, $P_{0_l,i+2}, \ldots, P_{0_l,i+t}$ $(l = 1, 2, \ldots, s)$ and $P_{i+j,i+k}$ for $1 \leq j, k, \leq t$ and $j \neq k$. This is a $(t \oplus s)$-clique inequality. We have n of these subconfigurations one for each $i = 1, 2, \ldots, n$. Adding up the inequalities of Lemma 5.7.3 for each of these subconfigurations gives

$$n\sum_{i=1}^{s} x_{0_i} + t(t-2+s)\sum_{v\in\mathcal{E}} x_v + t\sum_{v\in\mathcal{O}} x_v + t\sum_{v\in\mathcal{S}} x_v + \sum_{i=1}^{t-1}(t-i)\sum_{v\in\mathcal{R}_i} x_v$$
$$\leq n + t\frac{|\mathcal{S}|}{2} + \sum_{i=1}^{t-1}(t-i)\frac{|\mathcal{R}_i|}{2} + \frac{t(t-3+s)}{2}|\mathcal{E}|$$

where $\mathcal{R}_i$ is the set of internal vertices of paths corresponding to the cross-edges of type i. For each path P arising from a cross-edge of type i, we define the set $A_i(P)$ as in the proof of Theorem 5.5.2 and add the following i times: $x_u + x_v \leq 1$ for every $\{u, v\} \in A_i(P)$. Hence the resulting inequality is

$$n\sum_{i=1}^{s} x_{0_i} + \left(t(t-2+s) + 2\sum_{i=1}^{t-1} i\right)\sum_{i\in\mathcal{E}} x_i$$
$$+ t\sum_{i\in\mathcal{O}} x_i + t\sum_{v\in\mathcal{S}} x_v + \sum_{i=1}^{t-1}(t-i+i)\sum_{v\in\mathcal{R}_i} x_v$$
$$\leq n + t\frac{|\mathcal{S}|}{2} + \sum_{i=1}^{t-1}(t-i+i)\frac{|\mathcal{R}_i|}{2} + \frac{t(t-3+s)}{2}|\mathcal{E}| + 2\left(\sum_{i=1}^{t-1} i\right)\frac{|\mathcal{E}|}{2}.$$

This simplifies to

$$n\sum_{i=1}^{s} x_{0_i} + t(2t-3+s)\sum_{i\in\mathcal{E}} x_i + t\sum_{i\in\mathcal{O}} x_i + t\sum_{v\in\mathcal{S}\cup\mathcal{R}} x_v$$
$$\leq n + t\frac{|\mathcal{S}| + |\mathcal{R}| + (2t-4+s)|\mathcal{E}|}{2}.$$

After we divide the inequality by t we notice, that by Lemma 5.7.6 the term $(|\mathcal{S}|+|\mathcal{R}|+(2t-4+s)|\mathcal{E}|)/2$ is integral. So we can first round down the coefficients of the hub vertices and then the right hand side. Hereby the desired inequality is obtained:

$$\left\lfloor \frac{n}{t} \right\rfloor \sum_{i=1}^{s} x_{0_i} + (2t+s-3)\sum_{i\in\mathcal{E}} x_i + \sum_{i\in\mathcal{O}} x_i + \sum_{v\in\mathcal{S}\cup\mathcal{R}} x_v \leq \left\lfloor \frac{n}{t} \right\rfloor + \frac{|\mathcal{S}|+|\mathcal{R}|+(2t-4+s)|\mathcal{E}|}{2}. \qquad \square$$

We note that we can extend all these inequalities to include the inequalities for the corresponding nonsimple configurations by Lemma 5.2.1. As before, such inclusion will prove crucial for our separation algorithm. But before we can present the algorithm we need a path-representation of the inequality of Theorem 5.7.5.

Lemma 5.7.7.
An antiweb-s-wheel inequality $I_{\mathcal{AWW}}$ can be written as

$$-\sum_{P\in\mathcal{P}(S)} w(P) - \sum_{i=1}^{t-1}\sum_{P\in\mathcal{P}(J_i)} w(P) - (2t-4+s)\sum_{v\in\mathcal{E}\cup\mathcal{O}} f(v) - \left(\frac{n}{2} - \left\lfloor \frac{n}{t} \right\rfloor\right)\sum_{i=1}^{s} x_{0_i} + \left(\frac{n(s+2)}{4} - \left\lfloor \frac{n}{t} \right\rfloor\right) \leq 0$$

where $\mathcal{P}(S)$ and $\mathcal{P}(J_i)$ are the sets of paths derived from the spokes and i-edges, respectively.

PROOF. It is clear that we only have to prove this result for the case where the antiweb-s-wheel is simple, the nonsimple case follows directly. Consider the multivariable function

$$\phi = -\sum_{P\in\mathcal{P}(S)} w(P) - \sum_{i=1}^{t-1}\sum_{P\in\mathcal{P}(J_i)} w(P) - (2t-4+s)\sum_{v\in\mathcal{E}\cup\mathcal{O}} f(v) - \left(\frac{n}{2} - \left\lfloor \frac{n}{t} \right\rfloor\right)\sum_{i=1}^{s} x_{0_i} + \left(\frac{n(s+2)}{4} - \left\lfloor \frac{n}{t} \right\rfloor\right)$$

and compare the coefficients of the terms.

1. $[x_{0_i}]\phi$: We have $[x_{0_i}]\phi = [x_{0_i}](-\sum_{P\in\mathcal{P}(S)} w(P)) - (n/2 - \lfloor n/t \rfloor) = n/2 - n/2 + \lfloor n/t \rfloor = \lfloor n/t \rfloor$.

2. $[x_v]\phi$ *where* v *is an internal vertex in* $P \in \mathcal{P}(S) \cup \mathcal{P}(J_1) \cup \cdots \cup \mathcal{P}(J_t)$*:* As the degree of x_v is two and x_v appears in either $-\sum_{P\in\mathcal{P}(S)} w(P)$ or $-\sum_{i=1}^{t-1}\sum_{P\in\mathcal{P}(J_i)} w(P)$ only, $[x_v]\phi = 1/2 + 1/2 = 1$.
3. $[x_v]\phi$ *where* $v \in \mathcal{E}$*:* Then we obtain $[x_v](-\sum_{P\in\mathcal{P}(S)} w(P)) = s/2$, $[x_v](-\sum_{i=1}^{t-1}\sum_{P\in\mathcal{P}(J_i)} w(P)) = (2t-2)/2$ and $[x_v]f(v) = -1/2$. Hence $[x_v]\phi = s/2 + (2t-2)/2 + (2t-4+s)/2 = 2t-3+s$.
4. $[x_v]\phi$ *where* $v \in \mathcal{O}$*:* The first and second term are the same as in the last case, for the third term we obtain $[x_v]f(v) = 1/2$. Hence $[x_v]\phi = s/2 + (2t-2)/2 - (2t-4+s)/2 = 1$.
5. $[1]\phi$*:* We note that $[1](\sum_{P\in\mathcal{P}(S)} w(P) + \sum_{i=1}^{t-1}\sum_{P\in\mathcal{P}(J_i)} w(P))$ is equal to $1/2$ times the number of edges in the antiweb-s-wheel. The number of edges in the antiweb-s-wheel is $sn + n(t-1) + |\mathcal{S}| + |\mathcal{R}| = n(s+t-1) + |\mathcal{S}| + |\mathcal{R}|$. Moreover, we observe that $[1]f(v) = 1/4$ if $v \in \mathcal{E}$ and $[1]f(v) = -1/4$ if $v \in \mathcal{O}$. Hence
$$[1]\phi = -\frac{n(s+t-1)+|\mathcal{S}|+|\mathcal{R}|}{2} - \frac{2t-4+s}{4}(|\mathcal{E}|-|\mathcal{O}|) + \frac{n(s+2)}{4} - \left\lfloor\frac{n}{t}\right\rfloor = -\left\lfloor\frac{n}{t}\right\rfloor - \frac{|\mathcal{S}|+|\mathcal{R}|+(2t-4+s)|\mathcal{E}|}{2}$$
as $|\mathcal{O}| = n - |\mathcal{E}|$. □

Corollary 5.7.8.
Let W determine an inequality of the form $I_{\mathcal{AWW}}$ that is most-violated by x^*. Then every path in $\mathcal{P}(S)\cup\mathcal{P}(J_1)\cup\mathcal{P}(J_2)\cup\cdots\cup\mathcal{P}(J_{t-1})$ is a minimum-weight walk with respect to w^* of its parity joining its ends. In other words, if P is such a (nonempty) walk, and P joins a to b, then for every nonempty walk Q from a to b having the same parity as P, the weight of Q is at least the weight of P with respect to w^*.

PROOF. With
$$F = \sum_{P\in\mathcal{P}(S)} w(P) + \sum_{i=1}^{t-1}\sum_{P\in\mathcal{P}(J_i)} w(P) + (2t-4+s)\sum_{v\in\mathcal{E}\cup\mathcal{O}} f(v) + \left(\frac{n}{2}-\frac{n}{t}\right)\sum_{i=1}^{s} x_{0_i} - \left(\frac{n(s+2)}{4} - \frac{n}{t}\right)$$
we rewrite $I_{\mathcal{AWW}}$ as $F \geq \left(\frac{n}{t} - \left\lfloor\frac{n}{t}\right\rfloor\right)(1 - \sum_{i=1}^{s} x_{0_i})$ (from Lemma 5.7.7). □

Motivated by Corollary 5.7.8 we define the auxiliary graph H and the weight-functions w_E^*, w_O^* and w^+ as in the proof of Lemma 5.6.2 and proceed now with the construction of the separation method. Notice that it suffices for separation, to solve the problem with a fixed hub-set $\{v_{0_1}, v_{0_2}, \ldots, v_{0_s}\}$.

For the ordered t-clique (possibly degenerate) $(u_1, u_2, \ldots, u_t)$, we assign the following weight to it:

$$\begin{aligned}\frac{1}{t}\sum_{l=1}^{s}\sum_{i=1}^{t} w^+(v_{0_l}^{\mathcal{O}}, u_i) + \sum_{1\le j<i\le t}\left(\frac{1}{t-(i-j)}\right) w^+(u_i,u_j)\\ -\frac{s+2}{4}+\frac{1}{t}+\frac{t-2}{2t}\sum_{l=1}^{s} x^*_{0_l}\\ +\frac{t-3+s}{t}\sum_{i=1}^{t} f^+(u_i) + \sum_{1\le j<i\le t}\left(\frac{1}{t-(i-j)}-\frac{1}{t}\right)\left(f^+(u_i)+f^+(u_j)\right).\end{aligned}$$

We claim that this weight is nonnegative. We rewrite it as

$$\begin{aligned}\frac{1}{t}\Bigg(\sum_{l=1}^{s}\sum_{i=1}^{t} w^+(v_{0_l}^{\mathcal{O}}, u_i) + (t-3+s)\sum_{i=1}^{t} f^+(u_i)\\ +\frac{t-2}{2}\sum_{l=1}^{s} x^*_{0_l} + \sum_{1\le j<i\le t} w^+(u_i,u_j) - \frac{t(s+2)}{4}+1\Bigg)\\ +\sum_{1\le j<i\le t}\left(\frac{1}{t-(i-j)}-\frac{1}{t}\right)(w^+(u_i,u_j)+f^+(u_i)+f^+(u_j)).\end{aligned}$$

The first two lines together are nonnegative because that term is proportional to the form given in Lemma 5.7.4 and x^* satisfies the trivial, edge, cycle and $(t \oplus s)$-clique inequalities. Each summand in the third line is nonnegative by Lemma 5.6.1.

Suppose we have n ordered t-cliques, $(u_1^1, u_2^1, \ldots, u_t^1), (u_1^2, u_2^2, \ldots, u_t^2), \ldots, (u_1^n, u_2^n, \ldots, u_t^n)$, such that $u_{j+1}^i = u_j^{i+1}$ and $u_1^i \neq u_t^{i+1}$ for $i = 1, 2, \ldots, n-1$ and $j = 1, 2, \ldots, t-1$, and $u_j^n = u_{j+1}^1$ for $j = 1, 2, \ldots, t-1$ and $u_t^n \neq u_1^1$. We note that these t-cliques generate a possibly degenerate (n,t)-antiweb. Unlike the proof of Theorem 5.3.2, a degenerate (n,t)-antiweb poses no problem as a loop induces a nonempty walk in the original graph.

However, they may induce a nonsimple (n,t)-antiweb-1-wheel in the original graph. Moreover, the total weight of these n ordered t-cliques is

$$\sum_{P\in\mathcal{P}(S)} w^+(P) + \sum_{i=1}^{t-1}\sum_{P\in\mathcal{P}(J_i)} w^+(P) + (2t-4+s)\sum_{v\in\mathcal{E}\cup\mathcal{O}} f^+(v) \\ + \left(\frac{n}{2}-\frac{n}{t}\right)\sum_{l=1}^{s} x^*_{0_l} - \left(\frac{n(s+2)}{4}-\frac{n}{t}\right).$$

To see this, we first observe that we only have to prove the claim in which the resulting (n,t)-antiweb is simple; the nonsimple case follows immediately. We proceed as follow.

1. For a given u_i and $1 \le l \le s$ the term $w^+(v^{\mathcal{O}}_{0_l}, u_i)$ appears with weight $\frac{1}{t}$ in each of the exactly t of the n t-tuples; this gives $w^+(v^{\mathcal{O}}_{0_l}, u_i)$ for the total contribution.
2. For a given u_i and u_j, with $j < i$, where (u_i, u_j) is a k-edge in the generated (n,t)-antiweb. Then it appears in $t-k$ of the t-tuples; this gives $w^+(u_i, u_j)$ for the total contribution as $k = i - j$.
3. Let $u \in \mathcal{E}\cup\mathcal{O}$. Then u appears in exactly t of the n t-tuples. Moreover, it appears as the ith entry of a t-tuple exactly once for every $1 \le i \le t$. Hence the total contribution is $((t-3+s)/t\sum_{i=1}^{t} 1 + 2\sum_{1\le j<i\le t}(1/(t-(i-j))-1/t))f^*(u)$ and that equals $(2t-4+s)f^*(u)$.
4. As $x^*_{0_l}$ appears with weight $(t-2)/(2t)$ in each of the n t-tuples, this gives $\left(\frac{n}{2}-\frac{n}{t}\right)x^*_{0_l}$ for the total contribution.
5. The constant term in each of the n t-tuples is $-(s+2)/4+\frac{1}{t}$ so the total contribution is $-(n(s+2))/4 + n/t$

Hence the claim is established.

This gives a violated inequality for the corresponding antiweb-s-wheel if and only if this value is less than $\left(\frac{n}{t}-\left\lfloor\frac{n}{t}\right\rfloor\right)(1-\sum_{l=1}^{s} x^*_{0_l})$. So we have proved the next lemma.

Lemma 5.7.9.
If A is a minimum-weight (t)-antiweb in H of size congruent to $q \bmod t$, then x^* satisfies all (n,t)-antiweb-s-wheel inequalities with n congruent to $q \bmod t$ and hub-set $\{v_{0_1}, v_{0_2}, \ldots, v_{0_s}\}$ if and only if the weight of A is at least $\left(\frac{n}{t}-\left\lfloor\frac{n}{t}\right\rfloor\right)(1-\sum_{i=1}^{s} x^*_{0_i})$.

Hence we want to find such a minimum weight (t)-antiweb (possibly degenerate) generated by these t-cliques in H. The next corollary is obtained by applying Theorem 5.3.1 for each $q = 1, 2, \dots, t-1$ and every possible hub-set $\{v_{0_1}, v_{0_2}, \dots, v_{0_s}\}$ to Lemma 5.7.9.

Theorem 5.7.10.
For fixed s and t holds that the separation problem for (t)-antiweb-s-wheel inequalities with respect to $Q'_{(t+s-1)\oplus 1}STAB(G)$ can be solved in polynomial time.

5.8. Strength of the Antiweb Inequalities

An important question regarding the separability of the antiweb inequalities is whether they are a subset of a larger but instead polynomially separable class of cuts. One well-known large and polynomially separable class of cuts is the set of orthogonality constraints as introduced in [GLS93]. In this section we will prove that the antiweb inequalities are not implied by the orthogonality constraints.

Another important measure of the strength of a class of inequalities is the question whether they are facet-inducing and new. That antiweb inequalities are facet-inducing is shown in [Lau89]. For the question of novelty it is interesting to verify, whether the antiweb inequalities belong to the noncombinatorial class of orthogonality constraints. This will be answered to the negative in this section.

Definition 5.8.1.
Given a graph $G(V, E)$, an assignment of vectors a_v from some $\mathbb{R}^d$ to the vertices $v \in V$ is an orthogonal labeling *of G if for every pair u, v of nonadjacent vertices holds $a_u \cdot a_v = 0$.*

Orthogonal labelings are a way to generalize the polyhedral formulation of the stable set problem of $\bar{G}$. Whenever there is an edge $\{u, v\}$ in $\bar{G}$ (that is whenever u, v are not adjacent in G) we require for the stable set polytope of $\bar{G}$ that

$$x_u + x_v \leq 1. \tag{5.1}$$

If, for the purpose of this paragraph, we restrict the dimension d of the orthogonal labeling to 1 we see that for binary labelings x the inequality (5.1) can be as well described by the equation

$$x_u \cdot x_v = 0. \tag{5.2}$$

In a way this correspondence establishes, that orthogonal labelings are just a higher-dimensional analogue of stable sets of $\bar{G}$. Another interpretation is the following: as stable sets of $\bar{G}$ are cliques of G, orthogonal labelings are a higher-dimensional analogue of cliques of G.

Next we can define the cost c of a vector a_v of an orthogonal labeling a of G by 0 if $a_v = 0$ and otherwise by

$$c(a_v) = \frac{a_{v1}^2}{a_{v1}^2 + \cdots + a_{vd}^2}.$$

As QSTAB(G) is the set of all $x \in \mathbb{R}_+^V$ that fulfill for all cliques Q of G that $\sum_{v \in A} x_v \leq 1$ one can define similarly the set $TH(G)$ to be the set of all $x \in [0,1]^V$ that fulfill for all orthogonal labelings a of G the constraints $a^T x \leq 1$ (TH was in fact first defined in [GLS93]). This construction raises of course the question of the relation between $TH(G)$ and STAB(G) and QSTAB(G). The next proposition settles this.

Proposition 5.8.2 ((9.3.4) of [GLS93])**.**

$$\mathit{STAB}(G) \subseteq TH(G) \subseteq \mathit{QSTAB}(G)$$

Similar to the well-known *weighted stability number*

$$\alpha(G, w) = \max\{w^T x \colon x \in \mathrm{STAB}(G)\}$$

one can define the *Lovász-Theta function* that is originally defined in [Lov79]; but we stress here, that the $\vartheta(G)$ given in [Lov86] corresponds here and in the remaining literature to $\vartheta(\bar{G})$. So $\vartheta(G, w) = \max\{w^T x \colon x \in TH(G)\}$; similarly we use $\vartheta(G)$ as an abbreviation for $\vartheta(G, \mathbf{1})$. A simple consequence of Proposition 5.8.2 is

$$\alpha(G, w) \leq \vartheta(G, w) \leq \max\{w^T x \colon x \in \mathrm{QSTAB}(G)\}.$$

The *orthogonality constraints* are now all inequalities of the form $w^T x \leq \vartheta(G, w)$ for $w \in \mathbb{R}_+^V$. Of course they are valid inequalities for STAB(G); furthermore their intersection (with the positive orthant) is $TH(G)$, see [GLS93, Cor. 9.3.22(b)] and they prove that $TH(G)$ is a polytope if and only if G is perfect. As we want to show that $TH(G) \setminus \mathrm{A}_t\mathrm{STAB}(G) \neq \emptyset$ it suffices to show for a family of graphs G_l that $\vartheta(G_l) > \alpha_{\mathrm{A}_t\mathrm{STAB}}(G_l)$ where $\alpha_{\mathrm{A}_t\mathrm{STAB}}(G_l) = \max\{\mathbf{1}^T x \colon x \in \mathrm{A}_t\mathrm{STAB}(G_l)\}$.

For our proof we will need another (equivalent) definition of $\vartheta(w, G)$ (this is the function ϑ_4 of [Knu94]; there it is shown in Theorem 12, that

ϑ and ϑ_4 are the same). Here is the definition of ϑ_4 :

$$\vartheta_4(G, w) = \max \left\{ \sum_v c(b_v) w_v \colon b \text{ is an orthogonal labeling of } \bar{G} \right\}$$

Because we only aim to prove that $\vartheta(G_l) > \alpha_{\mathrm{A}_t\mathrm{STAB}}(G_l)$ we can make the next two simplifying assumptions:

1. As we know $\alpha_{\mathrm{A}_t\mathrm{STAB}}(G_l)$ exactly for G_l being a (t)-antiweb, it suffices to provide a lower bound on $\vartheta(G_l, \mathbf{1})$. So to bound ϑ_4 from below we need to find only a *good* orthogonal labeling of $\bar{G}_l$.
2. It suffices to prove $\vartheta(G_l) > \alpha_{\mathrm{A}_t\mathrm{STAB}}(G_l)$ for (sufficiently) large l.

Therefore, for G_s we will consider only the $(6s+1, 3)$-antiwebs. So, let $n = 6s + 1$ and $d = 2s + 1$.

We want to obtain an orthogonal labeling of $\overline{\mathcal{AW}(n, 3)}$. We start with the approach

$$b_v = \begin{pmatrix} \alpha_0 \\ \alpha_1 \cos(\varphi v) \\ \alpha_1 \sin(\varphi v) \\ \alpha_2 \cos(\varphi d v) \\ \alpha_2 \sin(\varphi d v) \end{pmatrix}, \tag{5.3}$$

where α_1, α_2 and φ are parameters to be determined.

For an orthogonal labeling of $\overline{\mathcal{AW}(n, 3)}$ we need that $b_v \cdot b_{v+1} = 0 = b_v \cdot b_{v+2}$ for all v. The choice $\varphi = \frac{n-1}{n}\pi$ guarantees that $m\varphi$ is a multiple of 2π iff m is a multiple of n (for this the fact $n \equiv 1 \bmod 2$ is used). Notice that

$$\begin{aligned} 0 = b_v \cdot b_{v+1} &= \alpha_0^2 + \alpha_1^2 \cos(\varphi v) \cos(\varphi(v+1)) + \alpha_1^2 \sin(\varphi v) \sin(\varphi(v+1)) \\ &\quad + \alpha_2^2 \cos(\varphi d v) \cos(\varphi d(v+1)) + \alpha_2^2 \sin(\varphi d v) \sin(\varphi d(v+1)). \end{aligned}$$

Now use the identity $\cos(\beta)\cos(\gamma) + \sin(\beta)\sin(\gamma) = \cos(\beta - \gamma)$ from [BSMM99, Eq. (2.86)] to simplify the last expression to

$$0 = b_v \cdot b_{v+1} = \alpha_0^2 + \alpha_1^2 \cos(\varphi) + \alpha_2^2 \cos(d\varphi). \tag{5.4}$$

Similarly, the condition $0 = b_v \cdot b_{v+2}$ implies

$$0 = b_v \cdot b_{v+2} = \alpha_0^2 + \alpha_1^2 \cos(2\varphi) + \alpha_2^2 \cos(2d\varphi). \tag{5.5}$$

Subtracting the Equation (5.4) from Equation (5.5) leads to

$$(\cos(2\varphi) - \cos(\varphi))\, \alpha_1^2 + (\cos(2d\varphi) - \cos(d\varphi))\, \alpha_2^2 = 0. \tag{5.6}$$

Now we can use the trigonometric identity $\cos(\beta) - \cos(\gamma) = -2\sin\frac{\beta+\gamma}{2} * \sin\frac{\beta-\gamma}{2}$ from [BSMM99, Eq. (2.113)] to obtain

$$\alpha_1^2 = -\frac{\sin\left(\frac{3d}{2}\varphi\right)\sin\left(\frac{d}{2}\varphi\right)}{\sin\left(\frac{3}{2}\varphi\right)\sin\left(\frac{1}{2}\varphi\right)}\alpha_2^2.$$

To simplify matters, we fix α_2^2 to 1, because the cost of an orthogonal labeling is invariant under nonzero scalings. So we need to show that

$$\alpha_1^2 = -\frac{\sin\left(\frac{3d}{2}\varphi\right)\sin\left(\frac{d}{2}\varphi\right)}{\sin\left(\frac{3}{2}\varphi\right)\sin\left(\frac{1}{2}\varphi\right)} \tag{5.7}$$

is nonnegative. To prove this we first observe the following:

$$\begin{aligned}
\frac{d}{2}\varphi &= s\pi + \frac{2s}{6s+1}\pi = s\pi + \frac{1}{3}\varphi \\
\frac{3d}{2}\varphi &= 3s\pi + \frac{6s}{6s+1}\pi = 3s\pi + \varphi \\
d\varphi &= 2s\pi + \frac{4s}{6s+1}\pi = 2s\pi + \frac{2}{3}\varphi.
\end{aligned}$$

This implies for the trigonometric functions:

$$\begin{aligned}
\sin\left(\frac{3d}{2}\varphi\right)\sin\left(\frac{d}{2}\varphi\right) &= (-1)^{3s}\sin\varphi\,(-1)^s\sin\frac{\varphi}{3} \\
&= \sin\varphi\sin\frac{\varphi}{3} \\
\cos(d\varphi) &= \cos\left(\frac{2}{3}\varphi\right).
\end{aligned}$$

So we get the following simplified expression for α_1^2 :

$$\begin{aligned}
\alpha_1^2 &= -\frac{\sin\varphi\sin\left(\frac{1}{3}\varphi\right)}{\sin\left(\frac{1}{2}\varphi\right)\sin\left(\frac{3}{2}\varphi\right)} \\
&= -2\frac{\cos\left(\frac{1}{2}\varphi\right)\sin\left(\frac{1}{3}\varphi\right)}{\sin\left(\frac{3}{2}\varphi\right)}
\end{aligned}$$

by additionally using the identity $\frac{\sin\varphi}{\sin\left(\frac{1}{2}\varphi\right)} = 2\cos\left(\frac{1}{2}\varphi\right)$. Notice, that non-negativity of α_1^2 is easy, as:

- $0 < \frac{1}{2}\varphi < \frac{\pi}{2}$, hence $0 < \cos\left(\frac{1}{2}\varphi\right)$;
- $0 < \frac{1}{3}\varphi < \frac{\pi}{2}$, hence $0 < \sin\left(\frac{1}{3}\varphi\right)$; and
- $\pi < \frac{3}{2}\varphi < \frac{3}{2}\pi$, hence $0 > \sin\left(\frac{3}{2}\varphi\right)$.

So we can conclude that there is a solution with $\alpha_1^2 > 0$ and $\alpha_2^2 = 1$. It remains to show that α_0^2 also is nonnegative for this setting. From Equation (5.4) we obtain

$$\begin{aligned}\alpha_0^2 &= -\cos d\varphi + 2\cos\varphi \frac{\cos\left(\frac{1}{2}\varphi\right)\sin\left(\frac{1}{3}\varphi\right)}{\sin\left(\frac{3}{2}\varphi\right)} \\ &= -\cos\left(\frac{2}{3}\varphi\right) + 2\frac{\cos\varphi\cos\left(\frac{1}{2}\varphi\right)\sin\left(\frac{1}{3}\varphi\right)}{\sin\left(\frac{3}{2}\varphi\right)}.\end{aligned}$$

Now notice that $-\cos\frac{2}{3}\varphi > 0$, $\cos\varphi < 0$, $\cos\left(\frac{1}{2}\varphi\right) > 0$, $\sin\left(\frac{1}{3}\varphi\right) > 0$, and $\sin\left(\frac{3}{2}\varphi\right) < 0$. Hence, $\alpha_0^2 > 0$.

We obtain for $c(b_v)$ the following expression:

$$c(b_v) = \frac{-\cos\left(\frac{2}{3}\varphi\right) + 2\frac{\cos\varphi\cos\left(\frac{1}{2}\varphi\right)\sin\left(\frac{1}{3}\varphi\right)}{\sin\left(\frac{3}{2}\varphi\right)}}{1 - 2\frac{\cos\left(\frac{1}{2}\varphi\right)\sin\left(\frac{1}{3}\varphi\right)}{\sin\left(\frac{3}{2}\varphi\right)} - \cos\left(\frac{2}{3}\varphi\right) + 2\frac{\cos\varphi\cos\left(\frac{1}{2}\varphi\right)\sin\left(\frac{1}{3}\varphi\right)}{\sin\left(\frac{3}{2}\varphi\right)}}. \tag{5.8}$$

This can be simplified to

$$c(b_v) = \frac{-\cos\left(\frac{2}{3}\varphi\right) + 2\frac{\cos\varphi\cos\left(\frac{1}{2}\varphi\right)\sin\left(\frac{1}{3}\varphi\right)}{\sin\left(\frac{3}{2}\varphi\right)}}{1 - \cos\left(\frac{2}{3}\varphi\right) + 2(\cos\varphi - 1)\frac{\cos\left(\frac{1}{2}\varphi\right)\sin\left(\frac{1}{3}\varphi\right)}{\sin\left(\frac{3}{2}\varphi\right)}}. \tag{5.9}$$

The substitution $\psi = \frac{1}{6}\varphi$ yields

$$c(b_v) = \frac{-\sin 9\psi\cos 4\psi + 2\cos 6\psi\cos 3\psi\sin 2\psi}{(1-\cos 4\psi)\sin 9\psi + 2(\cos 6\psi - 1)\cos 3\psi\sin 2\psi}.$$

Next we set $x = \cos\psi$ and $y = \sin\psi$ to use the following formulas about trigonometric functions of multiple angles from [BSMM99, Eqs. (2.104), (2.105) using $1 = x^2 + y^2$]:

$$\begin{aligned}\sin 2\psi &= 2xy \\ \sin 9\psi &= 256yx^8 - 448yx^6 + 240yx^4 - 40yx^2 + y \\ \cos 2\psi &= x^2 - y^2 \\ \cos 3\psi &= 4x^3 - 3x \\ \cos 4\psi &= 8x^4 - 8x^2 + 1 \\ \cos 6\psi &= 32x^6 - 48x^4 + 18x^2 - 1\end{aligned}$$

and obtain

$$\begin{aligned} c(b_v) &= \frac{1}{32}\frac{(128x^8-288x^6+208x^4-48x^2+1)(16x^4-12x^2+1)}{(4x^6-9x^4+6x^2-1)x^2(16x^4-12x^2+1)} \\ &= \frac{1}{32}\frac{128x^8-288x^6+208x^4-48x^2+1}{(4x^6-9x^4+6x^2-1)x^2}. \end{aligned}$$

Recall that $x = \cos\left(\frac{1}{6}\frac{6s}{6s+1}\pi\right)$. Now everything is set to study the asymptotics of the lower bound of the theta function and the rank.

Lemma 5.8.3.

$$\begin{aligned} \lim_{s\to\infty}(6s+1)\cdot c(b_v) - 2s &= \lim_{s\to\infty} 2s\cdot(3c(b_v)-1) + c(b_v) \\ &= \frac{1}{3} + \lim_{s\to\infty} 2s\cdot(3c(b_v)-1) = \frac{1}{3} - \frac{\pi\sqrt{3}}{27} \approx 0.1318 \end{aligned}$$

PROOF. First, it is easy to check that $c(b_v) \to \frac{1}{3}$ as $s\to\infty$, as everything involved is continuous. The other term is more difficult, as $2s\to\infty$ but $(3c(b_v)-1)\to 0$; instead we can look at $\lim_{s\to\infty}\frac{3c(b_v)-1}{1/(2s)}$. This asks for l'Hospital's rule (see [HEU90, Thm. 50.1]), that is indeed applicable, because

- the numerator and denominator go to 0;
- the denominator is differentiable on $[1,+\infty[$ and its derivative is $\neq 0$ on $[1,+\infty[$;
- the numerator is differentiable on $[s_0,+\infty[$ for sufficiently large s_0.

The derivative of the denominator is $\frac{-1}{2s^2}$. For the derivative of the numerator we will use the chainrule to evaluate $\frac{d}{ds}x$ and $\frac{d}{dx}(3c(b_v)-1)$. The first term is simpler:

$$\frac{d}{ds}x = \frac{-\pi}{(6s+1)^2}\sin\frac{s\pi}{6s+1}.$$

For the second term we obtain

$$\frac{d}{dx}(3c(b_v)-1) = \frac{-3}{16}\cdot\frac{128x^8-208x^6+96x^4-11x^2+1}{x^3(4x^4-5x^2+1)(4x^6-9x^4+6x^2-1)}.$$

Hence, with l'Hospital we obtain:

$$\begin{aligned}
&\lim_{s\to\infty} \frac{3c(b_v)-1}{1/(2s)} \\
= \;&\lim_{s\to\infty} \frac{\frac{-\pi}{(6s+1)^2}\sin\frac{s\pi}{6s+1}\cdot\frac{-3}{16}\cdot\frac{128x^8-208x^6+96x^4-11x^2+1}{x^3(4x^4-5x^2+1)(4x^6-9x^4+6x^2-1)}}{\frac{-1}{2s^2}} \\
= \;&\frac{-3\pi}{16}\cdot\lim_{s\to\infty}\frac{2s^2}{(6s+1)^2}\cdot\lim_{s\to\infty}\sin\frac{s\pi}{6s+1} \\
&\cdot\lim_{s\to\infty}\frac{128x^8-208x^6+96x^4-11x^2+1}{x^3(4x^4-5x^2+1)(4x^6-9x^4+6x^2-1)} \\
= \;&\frac{-3\pi}{16}\cdot\frac{1}{18}\cdot\frac{1}{2}\cdot\frac{64}{9}\sqrt{3} \\
= \;&\frac{-\pi\sqrt{3}}{54} \\
\approx \;&-0.2015332628 \quad \square
\end{aligned}$$

The following theorem is now a simple consequence of the previous lemma.

Theorem 5.8.4.
For sufficiently large s it holds that $\vartheta(\mathcal{AW}(6s+1,3)) - \operatorname{rank}(\mathcal{AW}(6s+1,3)) > 0.1$. Hence the $(6s+1,3)$-antiweb inequalities do not belong to the class of orthogonality cuts.

Additionally, an empirical study verified that the $(6s+1,3)$-inequalities are not dominated by the orthogonality constraints for small to reasonable sized s ($s \leq 10000$).

5.9. General Applicable Subdivision Theorems

In this section we study three different procedures to obtain new facets of one graph from facets of a smaller, related graph. The first two procedures are known from the literature and therefore they are reviewed quickly. The third one is new. Finally their interactions are studied.

5.9.1. Adding an Apex. The operation of adding an apex and its polyhedral consequences was well studied, see [Chv75]. It turns out that adding an apex is the same as substituting the graph into one vertex of a K_2.

Definition 5.9.1 (apex and spokes).
Given a graph $G(V, E)$ and a vertex $v \notin V$ we define the graph G^v on the vertex set $V \cup \{v\}$ and edge set $E \cup \{\{u, v\}: u \in V\}$. The vertex v is called apex *of G^v. The edges build like $\{v_{0_i}, v\}$ with $v \in V$ of a graph $G^{v_{0_1}, v_{0_2}, \ldots, v_{0_k}}$ are called* spokes. *If G is an antiweb, then the edges of G in $G^{v_{0_1}, v_{0_2}, \ldots, v_{0_k}}$ are again referred to as* cross-edges *(that have naturally a type as defined for the antiweb).*

Remark 5.9.2.
Henceforth, whenever we speak of G^v we will assume that $v \notin V(G)$.

Proposition 5.9.3.
Given a graph G and an inequality $a^T x \le b$ which defines a facet of $STAB(G)$ with $b > 0$. Then the inequality $a^T x + b x_v \le b$ defines a facet of $STAB(G^v)$.

PROOF. The validity is immediate. From the facetness for STAB(G) we know that a $|V| \times |V|$ matrix Y exists, so that its column vectors belong to STAB(G) and span the old facet. Furthermore, all column vectors from Y belong also to the new facet and STAB(G^v). Another column vector which belongs to both is the vector e_v.

$$Y^{\text{new}} = \begin{pmatrix} & & & 0 \\ & Y & & \vdots \\ & & & 0 \\ 0 & \cdots & 0 & 1 \end{pmatrix}.$$

It is simple to see that Y^{new} has full rank. □

5.9.2. Double Edge Subdivision. The following is a special form of a result of Wolsey [WOL76].

Proposition 5.9.4 (double edge subdivision).
Let $G = (V, E)$ be a graph and $c^T x \le d$ $(c \ge 0,\ d > 0)$ be facet-inducing for $STAB(G)$ with $\{a, b\} \in E$ and $c_a \ge c_b = \gamma$. Let G' be the graph obtained from G by replacing $\{a, b\}$ by the path $a - y - z - b$ where $y, z \notin V$. Suppose that

1. *there exists a stable set S in G with $c^T x^S = d$ and $a, b \notin S$, and*
2. *there exists a stable set S in G with $c^T x^S = d$, $a \in S$, $b \notin S$ and $h \notin S$ for all $h \in N(b) \setminus \{a\}$.*

Then $c^T x + \gamma x_y + \gamma x_z \leq d + \gamma$ *is facet-inducing for* $STAB(G')$*.*

For a proof see [Che95, Lemma 2.3.6]. In fact, the converse of Proposition 5.9.4 is true. To see this, assume $I' : c^T x + \gamma x_y + \gamma x_z \leq d + \gamma$ is facet-inducing for STAB(G'). Then there exists a stable set S of G with $a, y \notin S$ such that its incidence vector satisfies I' with equality. Therefore, $z \in S$ and $b \notin S$. Hence $S' = S \setminus \{z\}$ is a stable set of G with $a, b \notin S'$ that satisfies $I : c^T x \leq d$ with equality. There also must exist a stable set S of G with $y, z \notin S$ whose incidence vector satisfies I' with equality. Hence $a, b \in S$ and all the neighbors of b in G' do not belong to S. Thus $S \setminus \{b\}$ satisfies Condition 2 in Proposition 5.9.4.

In almost all our applications, we have $\gamma = 1$. We note that if $c^T x \leq d$ is not an edge inequality, then Condition 1 in Proposition 5.9.4 is automatically satisfied. Furthermore, if $\deg(b) = 2$ then Condition 2 in Proposition 5.9.4 is also automatically satisfied.

The following lemma is the validity version of Proposition 5.9.4.

Lemma 5.9.5.
Let $G = (V, E)$ *be a graph and* $c^T x \leq d$ *(*$c \geq 0$*,* $d > 0$*) be valid for* $STAB(G)$ *with* $\{a, b\} \in E$ *and* $\gamma = \min\{c_a, c_b\}$*. Then* $c^T x + \gamma x_y + \gamma x_z \leq d + \gamma$ *is valid for* $STAB(G')$*.*

Proof. Consider a stable set S' of G' and its incidence vector $x^{S'}$. If $a, b \in S'$ then $y, z \notin S'$. Without loss of generality we can assume $c_a = \gamma$. Notice, that $S = S' - a$ is a stable set of G. Plugging x^S into the valid inequality of STAB(G) shows $c^T x^S \leq d$ and then $c^T x^{S'} + \gamma x_y^{S'} + \gamma x_z^{S'} = c^T x^S + \gamma \leq d + \gamma$. If only one of a, b is in S' then only one of y, z is in S'. Again, $S = S' \setminus \{y, z\}$ is stable in G and validity of the initial inequality of STAB(G) yields $c^T x^{S'} + \gamma x_y^{S'} + \gamma x_z^{S'} \leq c^T x^S + \gamma \leq d + \gamma$. □

We need the following Proposition from [BM94, Thm. 2.5].

Proposition 5.9.6.
Let $G = (V, E)$ *be a graph. Let* $a^T x \leq \alpha$ *be a facet-inducing inequality of* $STAB(G)$*. Suppose that* G *contains a path* $p - u - v - q$ *such that* u *and* v *are of degree 2. Assume also that* $a_p = a_u = a_v = \beta$*. Let* $G' = (V', E')$ *be the graph obtained from* G *by replacing the path by the edge* $\{p, q\}$*. Let* $\bar{a}_u = a_u$ *for* $u \in V'$ *and* $\bar{\alpha} = \alpha - \beta$*, then* $\bar{a}^T x \leq \bar{\alpha}$ *is facet-inducing for* $STAB(G')$*.*

Similarly, if only validity is required, the next lemma is helpful.

Lemma 5.9.7.
Let $G = (V, E)$ be a graph. Let $a^T x \leq \alpha$ be a valid inequality of $STAB(G)$. Suppose that G contains a path $p - u - v - q$ such that u and v are of degree 2. Assume also that $a_u = a_v = \beta \leq \min\{a_p, a_q\}$. Let $G' = (V', E')$ be the graph obtained from G by replacing the path by the edge $\{p, q\}$. Let $\bar{a}_u = a_u$ for $u \in V'$ and $\bar{\alpha} = \alpha - \beta$, then $\bar{a}^T x \leq \bar{\alpha}$ is valid for $STAB(G')$.

(Note that the condition is $a_u = a_v = \beta \leq \min\{a_p, a_q\}$ and not $a_u = a_v = \beta = \min\{a_p, a_q\}$.)

5.9.3. Star Subdivision. We give a new theorem for lifting valid inequalities and facets from a graph to another graph where all edges incident with a single vertex are subdivided once. This graph operation was already introduced in [BM94, Thm. 2.3]. But our operation leads to another class of faces for the new graph. The proof of validity is different from theirs. Surprisingly, the proof of the facetness is the same (repeated here for sake of completeness). Our result differs from that given in [BM94, Thm. 2.3] in that we relax one prerequisite of their theorem while we replace their other condition by a stronger requirement. The resulting theorem cannot be proved in the same generality as [BM94, Thm. 2.3]. A simple calculation demonstrates that if the incidence-structure for facetness of Theorem 5.9.8 has to be maintained and the inequality should be valid, then the generalizing parameter p of Theorem [BM94, Thm. 2.3] can only have value 1 for our theorem.

Theorem 5.9.8 (star subdivision)**.**
Let $G = (V, E)$ be a graph and $a^T x \leq \alpha$ be a nontrivial valid inequality. Let v be a vertex of G and $N = \{v_1, \dots, v_{k+1}\}$ be the set of neighbors of v where $k \geq 1$. Let $G' = (V', E')$ be the graph obtained from G by subdividing each edge $\{v, v_i\}$ with a new node v_i' for $i = 1, \dots, k+1$. Set
$\bar{a}_u = a_u$ for $u \in V \setminus \{v\}$,
$\bar{a}_v = k$,
$\bar{a}_{v_i'} = 1$ for $i = 1, 2, \dots, k+1$,
$\bar{\alpha} = \alpha + k$.

1. *Suppose that $a_v = 1$ (actually only $a_v > 0$ is necessary, but in this case the definition of $\bar{a}$ is less simple). Then $\bar{a}^T x \leq \bar{\alpha}$ is a valid inequality of $STAB(G')$.*

2. *If additionally $a^T x \leq \alpha$ defines a facet of $STAB(G)$ and for each $i = 1, \ldots, k+1$, there exists a stable set $\tilde{S}_i$ such that $a^T x^{\tilde{S}_i} = \alpha$ and $\tilde{S}_i \cap N = \{v_i\}$ then $\bar{a}^T x \leq \bar{\alpha}$ defines a facet of $STAB(G')$.*

Proof. For the first part suppose S' is a stable set that violates the new inequality:

1. If $v \in S'$ and the left hand side is $> \alpha + k$. Then v_i' is not in S' for all i. So $S' \setminus \{v\}$ violates the old inequality because the left hand side would be $> \alpha$.
2. If $v \notin S'$, v_i' in S' for all i and the left hand side is$> \alpha + k$. Then v_i is not in S' for all i. So $\{S' \setminus \{v_1', v_2', \ldots, v_{k+1}'\}\} \cup \{v\}$ is a stable set for the old graph. The left hand side is $> \alpha + k - (k+1) + 1 = \alpha$.
3. If $v \notin S'$, not all v_i' are in S' and the left hand side is $> \alpha + k$. Let U be the set of v_i' in S'. Then $|U| \leq k$ and $S' \setminus U$ is a stable set for old graph. The left hand side $> \alpha + k - |U| \geq \alpha$.

Now we consider the facetness. Notice that the old facet is spanned by a set of column vectors of the following form:

$$Y^{\text{old}} = \begin{pmatrix} 0 \cdots 0 & 0 \cdots 0 & 1 \cdots 1 \\ Y^{00} & Y^{01} & Y^{02} \\ Y^{10} & Y^{11} & Y^{12} \end{pmatrix}.$$

(The row indices are v_0 followed by $v_1, \ldots, v_{k+1}$ followed by the rest of the vertices.) These column vectors are incidence vectors of n stable sets $S_1, \ldots, S_n$ of G. As the incidence vectors corresponding to $\tilde{S}_1, \ldots, \tilde{S}_{k+1}$ are linearly independent, we can assume without loss of generality, that $S_i = \tilde{S}_i$ for $i = 1, 2, \ldots, k+1$ and that the columns of

$$\begin{pmatrix} 0 \cdots 0 \\ Y^{00} \\ Y^{10} \end{pmatrix}$$

correspond to the incidence vectors $\tilde{S}_i$ for $i = 1, 2, \ldots, k+1$.

For the new face consider for $i = 1, 2, \ldots, n$ the following sets:

$$S_i' = \begin{cases} (S_i \setminus \{v\}) \dot{\cup} \{v_1', v_2', \ldots, v_{k+1}'\}, & \text{if } v \in S_i, \\ S_i \dot{\cup} \{v\}, & \text{if } v \notin S_i, \end{cases}$$

and for $i = 1, 2, \ldots, k+1$:

$$S_{n+i}' \;=\; \tilde{S}_i \dot{\cup} (\{v_1', v_2', \ldots, v_{k+1}', \} \setminus \{v_i'\}).$$

Then the columns of

$$Y^{\text{new}} = \begin{pmatrix} 1 \cdots 1 & 1 \cdots 1 & 0 \cdots 0 & 0 \quad \cdots \quad 0 \\ 0 \cdots 0 & 0 \cdots 0 & & \\ \vdots \; \cdots \; \vdots & \vdots \; \cdots \; \vdots & \mathbf{E} & \mathbf{E}-\mathbf{I} \\ 0 \cdots 0 & 0 \cdots 0 & & \\ Y^{00} & Y^{01} & Y^{02} & Y^{00} \\ Y^{10} & Y^{11} & Y^{12} & Y^{10} \end{pmatrix}$$

correspond to stable sets in the new face. (The rows are indexed by $v, v_0, v_1, \dots, v_k, v_0', v_1', \dots, v_k'$ and then by all the other vertices.) Now assume that the face induced by $\bar{a}^T x \leq \bar{\alpha}$ is contained in the face induced by another inequality $b^T x \leq \bar{\alpha}$.

By subtracting the columns of the first block from those of the last block we see that $b_{v_i'} = \frac{b_v}{k}$ for $i = 1, 2, \dots, k+1$.

There is some number $\delta > 0$ such that $\bar{\alpha} - b_v = \delta\alpha$. To see this, note that $\delta < 0$ would imply $\bar{\alpha} < b_v$, but then the original inequality $b^T x \leq \alpha$ is invalid; $\delta = 0$ implies $\bar{\alpha} = b_v$, but, as the set $\{v, u\}$ is stable for all vertices u nonadjacent with v, this implies $b_u = 0$ for all non-neighbors u of v. Thereby, the graph induced by the non-zero-coefficients is a star, which is not 2-connected (the support of every facet inducing inequality can never have a complete subgraph as cut-set by [CHV75, Thm. 4.1]) unless some of $b_{v_i'}$'s are 0 and the support graph is the graph consisting of v or a graph consisting of an edge of the form $\{v, v_i'\}$. Both cases are impossible as we have a stable set that satisfies $\bar{a}^T x = \bar{\alpha}$ and hence $bx = \bar{\alpha}$ that does not contain v. Given v_i', we also have a stable set that satisfies $\bar{a}^T x = \bar{\alpha}$ and hence $b^T x = \bar{\alpha}$ that does not contain v and v_i'.

As $b^T x^{S_i'} = \bar{\alpha}$, $i = 1, 2, \dots, n$ this implies $b^T x^{S_i'} - b_v = \bar{\alpha} - b_v$, $i = 1, 2, \dots, n$. Now define a vector c by $c_u = b_u$ for all $u \in V \setminus \{v\}$ and $c_v = \frac{b_v}{k}$. As $b^T x^{S_i'} - b_v = c^T x^{S_i}$, and $\bar{\alpha} - b_v = \delta\alpha$, and a is the unique solution of $a^T x^{S_i} = \alpha$, $i = 1, 2, \dots, n$, (as the x^{S_i} are linearly independent) we can conclude that $c = \delta a$.

This in turn implies

$$b_u = \delta a_u \text{ for } u \in V \setminus \{v\} \text{ and } b_{v_i'} = \delta a_v \text{ and } b_v = \delta k a_v (= \delta k).$$

As $\alpha + k = \bar{\alpha}$ and $\bar{\alpha} = \delta\alpha + b_v = \delta\alpha + \delta k$ we can conclude $\delta = 1$. $\square$

Lemma 5.9.9 (Incidence vectors of star subdivided faces).
All stable sets S whose incidence vectors fulfill the valid inequality $\bar{a}^T x \leq \bar{\alpha}$ of Theorem 5.9.8 (constructed from a valid inequality of the not star subdivided underlying graph) with equality fulfill one of the following conditions
1. *$v \in S$,*
2. *$v \notin S$, but all $v_i' \in S$, or*
3. *$v \notin S$, and exactly k of the v_i' belong to S.*

Proof. Suppose for a contradiction that we have a stable set S with $v \notin S$ and at most $k-1$ of the v_i' belong to S. Then $(S \cup \{v\}) \setminus \{v_1', v_2', \ldots, v_{k+1}'\}$ is a stable set of the unsubdivided underlying graph but it violates that corresponding inequality. $\square$

Lemma 5.9.10 (Converse of Theorem 5.9.8).
The converse of Theorem 5.9.8 part 2 is true. That is, if one of the sets $\tilde{S}_i$ does not exist, then the resulting face is not a facet.

Proof. So assume there is no set $\tilde{S}$ with (say) $\tilde{S} \cap N = \{v_1\}$. Then we claim that the new face $\bar{a}^T x \leq \bar{\alpha}$ is contained in the face $x_v + x_{v_1'} \leq 1$. For this we will show that $\bar{a}^T x = \bar{\alpha}$ implies $x_v + x_{v_1'} = 1$.

Using the possible types for a stable set S' with $\bar{a}^T x^{S'} = \alpha$ characterized by Lemma 5.9.9 we notice that types 1 and 2 obviously fulfill $x_v + x_{v_1'} = 1$. So we need to study S' of type 3. A set S' of type 3 could contradict the equation $x_v + x_{v_1'} = 1$ only if v_1' does not belong to S'. If additionally $v_1 \notin S'$ then the stable set $S' \cup \{v_1'\}$ would violate $\bar{a}^T x \leq \bar{\alpha}$ (because $\bar{a}_{v_1'} = 1 > 0$).

So we can assume $v_1 \in S'$. If $S' \cap N$ were equal to $\{v_1\}$, then the set S' minus the v_i' vertices would belong to the facet $a^T x \leq \alpha$ and would intersect N only in $\{v_1\}$ contrary to the assumption that every stable set from the facet which contains v_1 contains another neighbor of v.

So we know, that S' contains at least two different v_i. Therefore S' can contain at most $k-1$ of the v_i'. So if we delete the (at most) $k-1$ different v_i' from S we obtain a stable set S with $a^T x^S \geq \bar{a}^T x^{S'} - (k-1) = \bar{\alpha} - (k-1) = \alpha + k - (k-1) = \alpha + 1$. So this beast would violate the valid inequality we started with. Hence it cannot be! $\square$

Lemma 5.9.11 (not Theorem 5.9.8).
If the valid inequality (with $a_v = 1$) to start with in Theorem 5.9.8 is not

a facet, then the resulting face is not a facet.

Proof. So assume the face $a^T x \leq \alpha$ of G is contained in another face $b^T x \leq \beta$. As $a_v = 1$ we can choose for $b^T x \leq \beta$ an inequality with $b_v = 1$. Then we want to show that the face $\bar{a}^T x \leq \bar{\alpha}$ is contained in $\bar{b}^T x \leq \bar{\beta}$. For this it suffices to show for every stable set S' of G' that $\bar{a}^T x^{S'} = \bar{\alpha}$ implies $\bar{b}^T x^{S'} = \bar{\beta}$.

Again we use the characterization of stable sets S' with $\bar{a}^T x^{S'} = \alpha$ by Lemma 5.9.9. For these cases we obtain:

1. $a^T x^{S'-v} = \alpha$ holds; hence follows $b^T x^{S'-v} = \beta$ and $\bar{b}^T x^{S'} = \bar{\beta}$.
2. $a^T x^{S'+v-N'} = \alpha$ holds; hence follows $b^T x^{S'+v-N'} = \beta$ and $\bar{b}^T x^{S'} = \bar{\beta}$.
3. notice that if $|S' \cap N'| < k$ then $a^T x^{S'-N'} > \bar{\alpha} - k = \alpha$, which is impossible. If $|S' \cap N'| > k$ then actually $|S' \cap N'| = k+1$ and S' is of second type. So we can assume $|S' \cap N'| = k$. Then $a^T x^{S'-N'} = \alpha$ hence $b^T x^{S'-N'} = \beta$ hence $\bar{b}^T x^{S'} = \bar{\beta}$.

$\square$

5.9.4. Interaction of Star Subdivision and Edge Subdivision.

Lemma 5.9.12 (Iterative application of Theorem 5.9.8)**.**
Assume that the vertices v, w of a graph G fulfill the conditions of Theorem 5.9.8 with respect to G and the facet defining inequality $a^T x \leq \alpha$. Let G' be the graph constructed as in Theorem 5.9.8 from v. Then w fulfills the conditions of Theorem 5.9.8 with respect to G' and $\bar{a}^T x \leq \bar{\alpha}$.

Proof. Let the neighbors of v in G be $\{v_1, v_2, \ldots, v_{k+1}\}$ and the neighbors of w in G are $\{w_1, w_2, \ldots, w_{l+1}\}$. Let $\tilde{S}_1, \tilde{S}_2, \ldots, \tilde{S}_{k+1}$ and $\tilde{U}_1, \tilde{U}_2, \ldots, \tilde{U}_{l+1}$ be the stable sets in the facet with $\tilde{S}_i \cap \{v_1, v_2, \ldots, v_{k+1}\} = v_i$ and $\tilde{U}_i \cap \{w_1, w_2, \ldots, w_{l+1}\} = w_i$.

There are two cases, depending on the adjacency of v and z.

Case v and w are non-adjacent in G: Let $N = \{v_1, v_2, \ldots, v_{k+1}\}$ be the set of neighbors of v in G and $M = \{w_1, w_2, \ldots, w_{l+1}\}$ the set of neighbors of w in G. Notice that $M' = M$ is additionally the set of neighbors of w in G', as v, w are non-adjacent.

For $i = 1, 2, \ldots, l+1$ let

$$\tilde{U}'_i = \begin{cases} \tilde{U}_i \setminus \{v\} \dot{\cup} \{v'_1, v'_2, \ldots, v'_{k+1}\} & : \text{if } v \in \tilde{U}_i, \\ \tilde{U}_i \dot{\cup} \{v\} & : \text{if } v \notin \tilde{U}_i. \end{cases}$$

Notice, that $a^T x^{\tilde{U}_i} = \alpha$ implies $\bar{a}^T x^{\tilde{U}'_i} = \bar{\alpha}$. Now we want to show that $\tilde{U}'_i$ fulfill the conditions of Theorem 5.9.8 with respect to $\bar{a}^T x \leq \bar{\alpha}$ and G'. Notice first $\bar{a}_w = a_w$ and $a_w = 1$ by assumption. Then consider $\tilde{U}'_i \cap M'$; by construction $w_i \in \tilde{U}_i$ and by case $w_i \neq v$, hence $w_i \in \tilde{U}'_i$ and $w_i \in \tilde{U}'_i \cap M'$ follows. Now assume that $w_j \in \tilde{U}'_i$. If $w_j \in \tilde{U}_i$, then $j = i$, otherwise either $w_j = v$ which is impossible or $w_j \in \{v'_1, v'_2, \ldots, v'_{k+1}\}$ which is also impossible. Hence we succeeded in proving $\tilde{U}'_i \cap M' = \{w_i\}$ and all conditions of Theorem 5.9.8 are satisfied.

Case v and w are adjacent in G: Without loss of generality assume that $v_1 = w$ and $w_1 = v$. Let $N = \{v_1, v_2, \ldots, v_{k+1}\}$ be the set of neighbors of v in G and $M = \{w_1, w_2, \ldots, w_{l+1}\}$ the set of neighbors of w in G. Notice that $M' = \{v'_1, w_2, w_3, \ldots, w_{l+1}\}$, the set of neighbors of w in G', is different from M.

For $i = 1, 2, \ldots, l+1$ let

$$\tilde{U}'_i = \begin{cases} \tilde{U}_i \setminus \{v\} \dot{\cup} \{v'_1, v'_2, \ldots, v'_{k+1}\} & : \text{if } v \in \tilde{U}_i, \\ \tilde{U}_i \dot{\cup} \{v\} & : \text{if } v \notin \tilde{T}_i. \end{cases}$$

Notice, that $a^T x^{\tilde{U}_i} = \alpha$ implies $\bar{a}^T x^{\tilde{U}'_i} = \bar{\alpha}$. Now we want to show that $\tilde{U}'_i$ fulfill the conditions of Theorem 5.9.8 with respect to $\bar{a}^T x \leq \bar{\alpha}$ and G'. Notice first $\bar{a}_w = a_w$ and $a_w = 1$ by assumption. As $\tilde{U}'_i = \tilde{U}_i \dot{\cup} \{v\}$ for $i = 2, 3, \ldots, l+1$ it follows that $\tilde{U}'_i \cap M' = \{w_i\}$ for $i = 2, 3, \ldots, l+1$. For $i = 1$ holds $\tilde{U}'_i = \tilde{U}_i \setminus \{v\} \dot{\cup} \{v'_1, v'_2, \ldots, v'_{k+1}\}$ and thereby $\tilde{U}'_1 \cap M' = \{v'_1\}$ and all conditions of Theorem 5.9.8 are satisfied.

□

Lemma 5.9.13.
Let H be a graph. Suppose H' is obtained from H by replacing an edge $\{a, b\}$ by $a - y - z - b$. Suppose $c^T x \leq d$ is not an edge inequality. If $c^T x \leq d$ is facet-inducing for STAB(H) with $c \geq 0$, $c_a \geq c_b = \gamma$ and $d > 0$, and $\{a, b\}$ satisfies Condition 2 of Proposition 5.9.4 with respect to $I : c^T x \leq d$, then $I' : c^T x + \gamma x_y + \gamma x_z \leq d + \gamma$ is facet-inducing for STAB(H'). Moreover, $\{a, y\}, \{y, z\}$ and $\{z, b\}$ satisfy Condition 2 of Proposition 5.9.4 with respect to I'; $(f, h) \in E(H) \cap E(H')$ satisfies Condition 2 of Proposition 5.9.4 with respect to I if and only if $\{f, h\}$ satisfies Condition 2 of Proposition 5.9.4 with respect to I'.

PROOF. First we prove the forward implication. It follows from Proposition 5.9.4 that I' is facet-inducing for STAB(H'). Clearly any edge of the path $a - y - z - b$ satisfies the hypotheses in Proposition 5.9.4 because of $\deg(y) = deg(z) = 2$. Now suppose $\{a_1, b_1\} \in E(H) \cap E(H')$. As $\{a_1, b_1\}$ (with respect to H) satisfies the hypotheses, there is a stable set of U whose incidence vector x^U satisfies $c^T x \leq b$ with equality and $a_1 \in U$, $b_1 \notin U$ and $f \notin U$ for all $f \in N(b_1) \setminus \{a_1\}$. If $b_1 \notin \{a, b\}$, then we can extend U to U' by letting $U' = (U \setminus \{z\}) \cup \{y\}$ if $a \notin U$, or by letting $U' = (U \setminus \{y\}) \cup \{z\}$ if $b \notin U$. Now U' does the job. If $b_1 = a$, then we can extend U to U' by letting $U' = (U \setminus \{y\}) \cup \{z\}$ (as $b \notin U$); therefore U' does the job. If $b_1 = b$, then we can extend s to s' by letting $U' = (U \setminus \{z\}) \cup \{y\}$ (as $a \notin U$); therefore U' does the job.

Now we do the backward implication. Suppose (f, h) satisfy Condition 2 of Proposition 5.9.4 with respect to I'. Let S be a stable set that fulfills the condition. Assume $c_f \geq c_h = \beta$. We consider several cases:

$a = h$: Then $b \neq f$, $f \in S$ and $h, b \notin S$. Because the incidence vector of S satisfies I' with equality, we may assume $y \in S$. Hence $S \setminus \{z\}$ satisfies Condition 2 of Proposition 5.9.4 with respect to I.

$b = h$: Then $a \neq f$, $f \in S$ and $h, a \notin S$. Because the incidence vector of S satisfies I' with equality, we may assume $y \in S$. Hence $S \setminus \{z\}$ satisfies Condition 2 of Proposition 5.9.4 with respect to I.

$h \neq a$ and $h \neq b$: (However one of a, b may be f.) We have to consider several subcases namely a) $a, b \in S$ and $y, z \notin S$, b) $a, z \in S$ and $y, b \notin S$ and c) $b, y \in S$ and $a, z \notin S$. It is easy to see that in each of these cases a feasible set for condition 2 is at hand. As S satisfies I' with equality, the only remaining cases are a) $y \in S$ and $a, b, z \notin S$ and b) $z \in S$ and $a, b, y \notin S$. Then $S \setminus \{y\}$ and $S \setminus \{z\}$ fulfill condition 2 in their respective cases. $\square$

Lemma 5.9.14 (Theorem 5.9.8 and Proposition 5.9.4 commute).
Assume that the vertex v of a graph G fulfills the conditions of Theorem 5.9.8 with respect to G and the facet defining inequality $a^T x \leq \alpha$. Furthermore assume, that the edge $\{u, w\}$ fulfills the assumptions of Proposition 5.9.4 with the particular set S and $a_u \geq a_w$ and $a^T x \leq \alpha$ is not the edge inequality $x_u + x_w \leq 1$.

1. *Let G' be the graph constructed as in Theorem 5.9.8 from v. Then G' fulfills the conditions of Proposition 5.9.4 for $\{u, w\}$ (if u, w are adjacent in G') or otherwise for $\{u, w'\}$ or $\{u', w\}$ with respect to $\bar{a}^T x \leq \bar{\alpha}$.*

2. *Let G'' be the graph constructed as in Proposition 5.9.4 from $\{u, w\}$ and $\bar{a}^T x \leq \bar{\alpha}$ the corresponding inequality. Then G'' fulfills the conditions of Theorem 5.9.8 for v with respect to $\bar{a}^T x \leq \bar{\alpha}$.*

Proof. For the first part we have to distinguish the three cases:
Case $u, w \neq v$: First notice that $\bar{a}_u = a_u \geq a_w = \bar{a}_w$. If $w \in N$ then $v \notin S$ and thereby $S \cup \{v\}$ does the job. If $w \notin N$ then depending on $v \in S$ or $v \notin S$ either $(S \setminus \{v\}) \cup \{v'_1, v'_2, \ldots, v'_{k+1}\}$ or $S \cup \{v\}$ does the job (as nothing in the neighborhood of w is changed).
Case $w = v$ (and $u \in N$): Say $u = v_i$. Let $w_{\text{new}} = v'_i$. Now notice that $\deg(w_{\text{new}}) = 2$. With the remark after Proposition 5.9.4 this shows, that the conditions of Proposition 5.9.4 are satisfied.
Case $u = v$ (and $w \in N$): Say $w = v_i$. Let $u_{\text{new}} = v'_i$. Notice that $1 = \bar{a}_{u_{new}} = a_u \geq a_w = \bar{a}_w$. Because of $u = v$ it follows $v \in S$. Now $(S \setminus \{v\}) \cup \{v'_1, v'_2, \ldots, v'_{k+1}\}$ does the job.
For the second part, notice first that the subdivision of $\{u, w\}$ with y, z (thereby creating the path $u - y - z - w$) cannot change the coefficient of v from 1 to anything wrong. So validity is already guaranteed. If u, w are not adjacent to v then the sets $\tilde{S}_i$ need to be augmented only with y or z to make them incident with the facet of $\bar{a}^T x \leq \bar{\alpha}$. If u and/or w are adjacent to v then still every set can be augmented with either y or z to keep it in the face $\bar{a}^T x \leq \bar{\alpha}$. If finally u or w is v (say $u = v$) then the sets $\tilde{S}_i$ that do not contain w can be augmented with z without disturbing the neighborhood of v while maintaining incidence with $\bar{a}^T x \leq \bar{\alpha}$. For sets $\tilde{S}_i$ that do contain w we can augment with y and the new set intersects the neighborhood of v only in y. □

The Lemmas 5.9.12, 5.9.13 and 5.9.14 are summarized by the next Theorem.

Theorem 5.9.15.
If for a graph $G = (V, E)$ with a facet inducing inequality $a^T x \leq b$, a set $W \subseteq V$ and a multiset $F \subseteq E$ are given so that at every vertex of W star subdivision could be applied and every edge of F could be doubly subdivided, then all of these feasible operations can be carried out in arbitrary order while creating at the same time a sequence of facet inducing inequalities for the intermediate graphs and the final graph. Notice however, that while doing these operations some edges in the intermediate graphs disappear. (As they might be either doubly subdivided, or they might be incident with

a vertex that is star subdivided.) But in the process of disappearing, they are replaced by an edge or vertex, that permits again subdivision. So it is only necessary, to rename elements of F and W while subdividing related vertices or edges.

5.9.5. Interaction of Adding an Apex and Other Operations. Next we study G^v with respect to the subdivision procedures if we know that they were applicable to G.

Lemma 5.9.16 (Lifting of Theorem 5.9.8).
Let $G = (V, E)$ be a graph and $a^T x \leq \alpha$ be a nontrivial valid inequality. Let v be a vertex of G and $N = \{v_1, \ldots, v_k\}$ be the neighbor set of v where $k \geq 1$. Let $G' = G^{v_{k+1}}$. Extend the vector a (according to Proposition 5.9.3) to G by $a'_u = a_u$ for all $u \in V$ and $a'_{v_{k+1}} = \alpha$.

1. *If $(G, v, a^T x \leq \alpha)$ fulfills the conditions of Theorem 5.9.8, part 1 then $(G', v, {a'}^T x \leq \alpha)$ does.*
2. *If $(G, v, a^T x \leq \alpha)$ additionally fulfills the conditions of Theorem 5.9.8, part 2 then $(G', v, {a'}^T x \leq \alpha)$ does.*

Furthermore, if $(G^{v_{k+1}}, v, {a'}^T x \leq \alpha)$ with $a'_{v_{k+1}} = \alpha$ fulfills the conditions of Theorem 5.9.8 part 1 and/or part 2 and ${a'}^T x \leq \alpha$ is not a 3-cycle inequality then $(G, v, a^T x \leq \alpha)$ does.

PROOF. Validity and facetness of ${a'}^T x \leq \alpha$ follow directly from Proposition 5.9.3. For part 1 there is nothing to show. For part 2 we know that for $i = 1, \ldots, k$, there exists a stable set $\tilde{S}_i$ such that $a^T x^{\tilde{S}_i} = \alpha$ and $\tilde{S}_i \cap N = \{v_i\}$. We set $\tilde{S}'_i = \tilde{S}_i$ for $i = 1, \ldots, k$ and $\tilde{S}'_{k+1} = \{v_{k+1}\}$. Obviously, the conditions of part 2 are fulfilled for $i = 1, \ldots, k$. For $i = k+1$ the facts ${a'}^T x^{\tilde{S}'_{k+1}} = \alpha$ and $\tilde{S}'_{k+1} \cap N = \{v_{k+1}\}$ are not more difficult.

For the other direction, note that part 1 is easy, while for part 2 it suffices, that all the $\tilde{S}'_i$ carry over from G' to G (except the one corresponding to v_{k+1}). □

Lemma 5.9.17 (Lifting of Proposition 5.9.4).
Let $G = (V, E)$ be a graph, $a^T x \leq \alpha$ be a nontrivial facet defining inequality and $\{u, v\}$ is an edge of G. If the condition 2 of Proposition 5.9.4 is fulfilled for $(G, a^T x \leq \alpha, u, v)$ then conditions 1 and 2 of Proposition 5.9.4 are also fulfilled for the graph $G' = G^w$, the inequality $a^T x = {a'}^T x + \alpha x_w \leq \alpha$ and the vertices u, v. Furthermore, if $(G^w, u, v, {a'}^T x \leq \alpha)$ with $a_w = \alpha$ fulfills

the conditions 1 and 2 of Proposition 5.9.4 and ${a'}^T x \leq \alpha$ *is not a* 3*-cycle inequality then* $(G, u, v, a^T x \leq \alpha)$ *does.*

Proof. Validity and facetness of ${a'}^T x \leq \alpha$ follows by Proposition 5.9.3 from these properties for $a^T x \leq \alpha$. Consider the stable set $S' = \{w\}$. Notice ${a'}^T x^{S'} = \alpha$ and $x_u^{S'} = x_v^{S'} = 0$. So condition 1 of Proposition 5.9.4 is satisfied. Condition 2 is verified for G' with the same set which fulfills it for G.

For the converse direction notice that the two required sets directly carry over from G' to G. □

Lemma 5.9.18 (subdivision of the spokes)**.**
Let $G = (V, E)$ *be a graph,* $a^T x \leq \alpha$ *be a facet defining inequality and* u *is a vertex of* G *so that the inequality is not* $x_u \leq 1$*. Then conditions 1 and 2 of Proposition 5.9.4 are also fulfilled for the graph* G^w *together with the inequality* ${a'}^T x = ax + \alpha x_w \leq \alpha$ *and the edge* $\{u, w\}$*.*

Proof. Validity and facetness are easy again. As $a'_w \geq a'_u$ the stable set $S = \{w\}$ does the job. Furthermore, the new inequality is not an edge inequality, because the old inequality was not the constraint $x_u \leq 1$. □

Lemma 5.9.19.
Let A *be an* (n, t)*-antiweb and* I_A *the corresponding inequality for* A*. Then all edges of type not greater than* $n \bmod t$ *fulfill condition (2) of Proposition 5.9.4. All other edges (those of type greater then* $n \bmod t$*) violate condition (2).*

Proof. Consider a rim-edge of type $d \leq (n \bmod t)$, say $\{v_1, v_{d+1}\}$ and let $f = \left\lfloor \frac{n}{t} \right\rfloor$. Obviously, $ft + d \leq n$. We claim that the set $S = \{v_1\} \cup \{v_{d+1+t}, v_{d+1+2t}, \ldots, v_{d+1+(f-1)t}\}$ is stable and fulfills condition (2) with respect to I. Regarding the stability it is easy to see that the second part of S is stable; it remains to verify that there is no edge between that part and v_1. But the latter is easy, as v_1 is adjacent only to its $t-1$ successors (and they are not in S) and the $t-1$ predecessors; for this notice, that the last element $v_{d+1+(f-1)t}$ of S could only be in conflict with with v_1 but the neighbor of $v_{d+1+(f-1)t}$ being closest to v_1 is the vertex v_{d+1+ft} which is—as a consequence of $ft+d \leq n$—distinct from v_1. Therefore S is stable. Regarding condition (2) notice that the neighbors of v_{d+1} which are not permitted to be in S are $\{v_{d+1-(t-1)}, v_{d+1-(t-2)}, \ldots, v_d\} \setminus \{v_1\} \cup \{v_0\} \cup$

$\{v_{d+2}, v_{d+3}, \dots, v_{d+t}\}$. The second and third part of the set are definitely not in S, for the vertices in the first part notice that if $d+1-(t-1) \leq 0$, then it corresponds to $d+1-(t-1)+n \geq (f-1)t+2d+2$ which is greater than $d+1+(f-1)t$ and if $d+1-(t-1) > 0$ (that is $d > t-2$) then follows together with $t > d$ that $d = t-1$ and $d+1-(t-1) = 1$, but we do not have to care about v_1.

Finally, consider a rim-edge of type $d > (n \bmod t)$ and set $f = \lfloor \frac{n}{t} \rfloor$, hence $ft+d > n$. Without loss of generality assume that the rim-edge is the edge $\{v_1, v_{d+1}\}$. We want to construct a stable set $S \ni v_1$ which fulfills additionally condition (2). Hence some vertices cannot be in S and the only choices besides v_1 for S belong to $\{v_{d+1+t}, v_{d+1+t+1}, \dots, v_{n+1-t}\}$ (the number of the last element is determined by the minimum distance of t from $v_1 = v_{n+1}$). We need to choose $f-1$ vertices; if there is a solution, then the remaining elements can be $\{v_{d+1+t}, v_{d+1+2t}, \dots, v_{d+1+(f-1)t}\}$. But the vertex $v_{d+1+(f-1)t}$ does not belong to the set of candidates, as $d+1+(f-1)t > n+1-t$. So this rim-edge of type $d > (n \bmod t)$ can never fulfill condition (2). □

Corollary 5.9.20.
Let A be an (n,t)-antiweb, k a nonnegative integer and $I_{A^{v_{0_1}, v_{0_2}, \dots, v_{0_k}}}$ the corresponding inequality for $G = A^{v_{0_1}, v_{0_2}, \dots, v_{0_k}}$. Then all edges of A of type not greater than $n \bmod t$ fulfill condition (2) of Proposition 5.9.4 with respect to G. All other edges of A (those of type greater than $n \bmod t$) violate condition (2) for G.

PROOF. The proof is done by induction on k. Lemma 5.9.19 establishes the base-case of $k = 0$. So assume that the theorem is proved for $(A^{v_{0_1}, v_{0_2}, \dots, v_{0_k}}, I_{A^{v_{0_1}, v_{0_2}, \dots, v_{0_k}}})$. Lemma 5.9.17 establishes the claim then for $(A^{v_{0_1}, v_{0_2}, \dots, v_{0_{k+1}}}, I_{A^{v_{0_1}, v_{0_2}, \dots, v_{0_{k+1}}}})$. □

Lemma 5.9.21.
Let G be an (n,t)-antiweb with $n \not\equiv 0 \bmod t$ and let v be a vertex of G. For the facet $\sum_{i=1}^{n} x_i \leq \lfloor \frac{n}{t} \rfloor$ the following two statements are equivalent :

1. *$n \equiv t-1 \bmod t$ and*
2. *v fulfills the assumptions 1 and 2 of Theorem 5.9.8.*

PROOF. Let $k = \lfloor \frac{n}{t} \rfloor$. For the first implication (hence $n = k*t+t-1$) we can assume without loss of generality (by symmetry) that $v = t$. Now we will construct the stable sets $\tilde{S}_i$ for all neighbors of i. Let $S' = \{3t -$

$1, 4t-1, \ldots, n\}$. Notice, $a^T x^S = k-1$ and S contains no neighbor of $t, t+1, \ldots, 2t-1$. Finally for i with $t < i < 2t$ the sets $\tilde{S}_i = S' \cup \{i\}$ are stable and fulfill $a^T x^{S_i} = k$. For the second halve of necessary sets consider $S'' = \{2t, 3t, \ldots, kt\}$ and use for i with $0 < i < t$ the sets $\tilde{S}_i = S'' \cup \{i\}$.

For the other direction again we can assume without loss of generality that vertex $v = t$. As the assumptions are fulfilled, there is a stable set $\tilde{S}$ with $S \cap \{1, 2, \ldots, 2t-1\} = \{2t-1\}$ and $a^T x^{\tilde{S}} = k$. So $\tilde{S}$ contains k elements. Choosing vertices of as small number as possible, $\tilde{S}$ must be $\{2t-1, 3t-1, \ldots, (k+1)t-1\}$. This requires that $n \geq (k+1)t-1$ and finally $n \equiv t-1 \bmod t$. □

Together, Lemmas 5.9.21 and 5.9.16 imply the next corollary.

Corollary 5.9.22.
Let A be an (n,t)-antiweb with $n \not\equiv 0 \bmod t$, let k be a nonnegative integer and let v be a vertex of A. Consider $A^{v_{0_1}, v_{0_2}, \ldots, v_{0_k}}$. For the facet $I_{A^{v_{0_1}, v_{0_2}, \ldots, v_{0_k}}} : \lfloor \frac{n}{t} \rfloor \sum_{i=1}^{k} x_{0_i} + \sum_{i=1}^{n} x_i \leq \lfloor \frac{n}{t} \rfloor$ of STAB(A^{v_0}) the following two statements are equivalent :

1. *$n \equiv t-1 \bmod t$ and*
2. *v fulfills the assumptions 1 and 2 of Theorem 5.9.8.*

Together, the results 5.9.16 – 5.9.22 of this subsection can be summarized by the next theorem.

Theorem 5.9.23.
Let A be an (n,t)-antiweb with $n \not\equiv 0 \bmod t$ and let k be a nonnegative integer. Let $G = A^{v_{0_1}, v_{0_2}, \ldots, v_{0_k}}$ and the corresponding inequality $I_{A^{v_{0_1}, v_{0_2}, \ldots, v_{0_k}}} : \lfloor \frac{n}{t} \rfloor \sum_{i=1}^{k} x_{0_i} + \sum_{i=1}^{n} x_i \leq \lfloor \frac{n}{t} \rfloor$. Then the following hold:

1. *Every spoke can be doubly subdivided.*
2. *Star subdivision at a vertex $v \in A$ is applicable in G if and only if $n \equiv t-1 \bmod t$.*
3. *An edge e of A can be doubly subdivided in G if and only if its type is at most $n \bmod t$.*

5.10. Facetness of Antiweb-Wheels

In this section we put together the composition results of the preceding sections to get a complete characterization of facet inducing proper antiweb-wheel inequalities. The inequalities of the graphs of type A^{v_0} of Section 5.9 are an example of this new class. For the reader's convenience

Definition 5.5.1 from Section 5.5 is reproduced here to facilitate easy reference.

Definition 5.10.1 (antiweb-1-wheel).
Given an (n, t)-antiweb $G_1 = (V_1, E_1)$ with $n \not\equiv 0 \bmod t$, given a partition $\mathcal{E}, \mathcal{O}$ of $V_1 = \{1, 2, \ldots, n\}$. Consider a subdivision G of $G_1^{v_0}$. Let $P_{0,i}$ denote the path obtained from subdividing the edge $\{v_0, v_i\}$ and let $P_{i,j}$ (for v_i, v_j adjacent in G_1) denote the path obtained from subdividing the edge $\{v_i, v_j\}$. This graph G is a simple antiweb-1-wheel *if the following four conditions are fulfilled:*

1. *the length of $P_{0,i}$ is even for $i \in \mathcal{E}$ and odd for $i \in \mathcal{O}$,*
2. *the length of the path $P_{i,j}$ is even for $i \in \mathcal{E}$ and $j \in \mathcal{O}$ or $j \in \mathcal{E}$ and $i \in \mathcal{O}$,*
3. *the length of the path $P_{i,j}$ is odd for $i, j \in \mathcal{O}$, and*
4. *the length of the path $P_{i,j}$ is odd for $i, j \in \mathcal{E}$.*

A simple antiweb-1-wheel is *proper* if additionally $P_{i,j}$ is of length at least 2 for all paths with at least one end in $\mathcal{E}$. A proper antiweb-1-wheel is *basic* with respect to a given partition $\mathcal{E}, \mathcal{O}$ if all the involved paths have minimal length.

Theorem 5.10.2 (validity of the (n, t)-antiweb-1-wheel inequality).
Given an (n, t)-antiweb-1-wheel G, the inequality

$$\left\lfloor \frac{n}{t} \right\rfloor x_0 + \sum_{i \in \mathcal{O}} x_i + (2t-2) \sum_{i \in \mathcal{E}} x_i + \sum_{v \in \mathcal{S} \cup \mathcal{R}} x_v \leq \left\lfloor \frac{n}{t} \right\rfloor + (2t-2)|\mathcal{E}| + \frac{|\mathcal{S}| + |\mathcal{R}| - (2t-1)|\mathcal{E}|}{2} \tag{5.10}$$

is valid for $STAB(G)$, where $\mathcal{S}$ denotes the set of internal vertices of the spoke-path and $\mathcal{R}$ denotes the set of internal vertices of the subdivided antiweb edges.

PROOF. As a starting point we utilize the validity of $I_{A^{v_0}} : \sum_{i=1}^{n} x_i + \lfloor \frac{n}{t} \rfloor \sum_{i=1}^{k} x_{0_i} \leq \lfloor \frac{n}{t} \rfloor$ for A^{v_0} from Theorem 5.9.23. Then we do star subdivision at all vertices of $\mathcal{E}$; here Theorem 5.9.8, Number 1 guarantees that validity is maintained. If in the antiweb-wheel we want to reach there are paths of length 1 between members of $\mathcal{E}$ then the paths of length 3 between them can be doubly contracted by Lemma 5.9.7. If finally between some spoke-ends longer paths are necessary, they can be produced by applying edge-subdivision, that maintains validity according to Lemma 5.9.5. ☐

Theorem 5.10.3 (proper antiweb-1-wheel facets for $n \equiv t - 1 \bmod t$)**.** *Given a proper* (n, t)*-antiweb-*1*-wheel* G *with* $n \equiv t - 1 \bmod t$ *then the inequality* (5.10) *induces a facet of* $STAB(G)$*.*

PROOF. We will first show that the theorem is true for basic (n, t)-antiweb-1-wheels. Given an (n, t)-antiweb-1-wheel H^{v_0} star subdivision is applicable to every vertex of H in H^{v_0} and double-edge subdivision is applicable to every edge of H in H^{v_0} with respect to

$$\left\lfloor \frac{n}{t} \right\rfloor x_0 + \sum_{i \in H} x_i \leq \left\lfloor \frac{n}{t} \right\rfloor \tag{5.11}$$

by Theorem 5.9.23. Furthermore, all spokes of H^{v_0} could be doubly subdivided.

By Theorem 5.9.15 we can carry out the star subdivision at all vertices of $\mathcal{E}$ in H^{v_0} with respect to inequality (5.11). This leads to the desired facet of the basic (n, t)-antiweb-1-wheel H'; the inequality can be written as

$$\left\lfloor \frac{n}{t} \right\rfloor x_0 + \sum_{i \in \mathcal{O}} x_i + (2t - 2) \sum_{i \in \mathcal{E}} x_i + \sum_{v \in \mathcal{S} \cup \mathcal{R}} x_v \leq \left\lfloor \frac{n}{t} \right\rfloor + (2t - 2)|\mathcal{E}|.$$

By Theorem 5.9.15 we can finally doubly subdivide the remaining edges (as necessary) and prove thereby the desired facet in general. ☐

Theorem 5.10.4 (proper antiweb-1-wheel facets ($1 \leq (n \bmod t) \leq t - 2$))**.** *Given a proper antiweb-*1*-wheel* G *with* $n \equiv a \bmod t$*,* $1 \leq a \leq t - 2$*. Then inequality* (5.10) *induces a facet of the corresponding stable set polytope if and only if*

1. $\mathcal{E} = \emptyset$ *and*
2. *all paths* $P_{i,j}$ *with* $\{i, j\}$ *of type* $> a$ *and* $i, j \in \mathcal{O}$ *have length* 1*.*

PROOF. The proof that 1 and 2 imply facetness is the same as that in Theorem 5.10.3, except that no star subdivision is done, and only edges of type $\leq n \bmod t$ are doubly subdivided (as necessary), while edges of type $> n \bmod t$ are not doubly subdivided (as they do not fulfill the prerequisites of double edge-subdivision).

For the forward direction consider an arbitrary but proper (n, t)-antiweb-wheel G' with partition $\mathcal{E} \dot\cup \mathcal{O}$ that is facet-inducing for $\text{STAB}(G')$. First we want to apply Proposition 5.9.6 to shorten paths of length 3 to edges while maintaining facetness in the following way:

- paths $P_{i,j}$ of type $> a$ and of length > 3 are shortened until their length is 2 or 3;
- paths $P_{i,j}$ of type $< a$ with both ends in $\mathcal{E}$ are shortened down to length 3;
- paths $P_{i,j}$ of type $< a$ with at most one end in $\mathcal{E}$ are shortened down to length 1 or 2;
- spoke-paths are shortened down to length 1 or 2.

Denote with G the resulting graph. Now if G' violates condition 1 or 2 then G does, as we did not change $\mathcal{E}$ at all and even though we shortened the subdivided paths, we took care that paths violating condition are not changed in any way.

Now consider a sequence of undoing star subdivisions and shortening paths of length 3 to edges to reduce G to an unsubdivided (n, t)-antiweb-wheel H. Notice, that by Lemma 5.9.11 the inequality after undoing a single star subdivision is again facet inducing. By Proposition 5.9.6 follows similarly that the inequality after shortening a path is facet inducing.

Now consider the last intermediate graph in this sequence H'. By Theorem 5.9.23 we know that neither star subdivision nor double-edge subdivision are applicable to H'. If the last operation is an star subdivision, then by Lemma 5.9.10 it follows that going from H to H' by star subdivision destroys facetness, that is the inequality for H' is not facet inducing contrary to assumption. If on the other hand the last operation is doubly subdividing an edge of type $> a$ then by the remark after Proposition 5.9.4 and the fact that we have a facet for H' we obtain that that edge does not violate condition (2) giving again a contradiction. $\square$

The construction of starting with an antiweb then adding an apex, doing star subdivision and then doubly subdividing the edges can be generalized by adding more (or no) apex. For the reader's convenience Definition 5.7.1 from Section 5.7 is reproduced here to facilitate easy reference.

Definition 5.10.5 (antiweb-s-wheel)**.**
Let $s \geq 0$. Given an (n, t)-antiweb $G_1 = (V_1, E_1)$ with $n \not\equiv 0 \bmod t$, given a partition $\mathcal{E}, \mathcal{O}$ of the set of vertices of $V_1 = \{1, 2, \ldots, n\}$. Consider a subdivision G of $G_1^{v_{0_1}, v_{0_2}, \ldots, v_{0_s}}$. Let $P_{0_i,j}$ denote the path obtained from subdividing the edge $\{v_{0_i}, v_j\}$ and let $P_{i,j}$ (for v_i, v_j adjacent in G_1) denote

the path obtained from subdividing the edge $\{v_i, v_j\}$. *This graph* G *is a* simple antiweb-s-wheel *if the following four conditions are fulfilled:*

1. *for all* $1 \leq i \leq s$ *is the length of* $P_{0_i,j}$ *even for* $j \in \mathcal{E}$ *and odd for* $j \in \mathcal{O}$,
2. *the length of the path* $P_{i,j}$ *is even for* $i \in \mathcal{E}$ *and* $j \in \mathcal{O}$ *or* $j \in \mathcal{E}$ *and* $i \in \mathcal{O}$,
3. *the length of the path* $P_{i,j}$ *is odd for* $i, j \in \mathcal{O}$, *and*
4. *the length of the path* $P_{i,j}$ *is odd for* $i, j \in \mathcal{E}$.

A simple antiweb-s-wheel is *proper* if additionally $P_{i,j}$ is of length at least 2 for all paths with at least one end in $\mathcal{E}$. A proper antiweb-s-wheel is *basic* with respect to a given partition $\mathcal{E}, \mathcal{O}$ if all the involved paths have minimal length.

Theorem 5.10.6 (validity of the (n,t)-antiweb-s-wheel inequality**).**
For $s \geq 0$ *and an* (n,t)*-antiweb-*s*-wheel* G *the inequality*

$$\left\lfloor \frac{n}{t} \right\rfloor \sum_{i=1}^{s} x_{0_i} + \sum_{i \in \mathcal{O}} x_i + (2t-3+s) \sum_{i \in \mathcal{E}} x_i + \sum_{v \in \mathcal{S} \cup \mathcal{R}} x_v \leq \left\lfloor \frac{n}{t} \right\rfloor + (2t-3+s)|\mathcal{E}| + \frac{|\mathcal{S}| + |\mathcal{R}| - (2t-2+s)|\mathcal{E}|}{2} \tag{5.12}$$

is valid for STAB(G), where $\mathcal{S}$ *denotes the set of internal vertices of the spoke-path and* $\mathcal{R}$ *denotes the set of internal vertices of the subdivided antiweb edges.*

PROOF. As a starting point, we utilize the validity of

$$\left\lfloor \frac{n}{t} \right\rfloor \sum_{i=1}^{k} x_{0_i} + \sum_{i=1}^{n} x_i \leq \left\lfloor \frac{n}{t} \right\rfloor \tag{$I_{A^{v_{0_1}, v_{0_2}, \ldots, v_{0_k}}}$}$$

for $A^{v_{0_1}, v_{0_2}, \ldots, v_{0_k}}$ from Theorem 5.9.23. Then we do star subdivision at all vertices of $\mathcal{E}$; here Theorem 5.9.8, Number 1 guarantees that validity is maintained. The degree of every vertex in $\mathcal{E}$ is $(2t-2)+s$, where the first term accounts for the neighbors in the antiweb and the second term for the neighborly hubs. So if star subdivision is applied at a vertex $v \in \mathcal{E}$ then $(2t-2)$ new rim vertices and s spoke vertices are added of weight 1. The weight of v is changed to $2t-3+s$ $(= \deg v - 1)$ and the right hand side is incremented by $2t-3+s$. This is accomplished in inequality (5.12) by the coefficient $2t-s+3$ of $\sum_{i \in \mathcal{E}} x_i$, the term $\sum_{v \in \mathcal{S} \cup \mathcal{R}} x_v$ and for the right hand side the term $(2t-3+s)|\mathcal{E}|$. Actually this operation subdivided also the spokes and cross-edges incident with v, thereby the term $|\mathcal{S}|+|\mathcal{R}|$ on the right hand side is increased by $(2t-2+s)|\mathcal{E}|$ so we need to subtract the same amount to balance this effect.

If in the antiweb-wheel we want to reach there are paths of length 1 between members of $\mathcal{E}$ then the paths of length 3 between them can be doubly contracted by Lemma 5.9.7. Every contraction step changes the inequality in that the two terms x_u and x_w corresponding to vertices contracted away are dropped on the left hand side of the inequality (as u, w are removed from $\mathcal{S}$ or $\mathcal{R}$) and at the same time the right hand side decreases by two.

If finally between some spoke-ends longer paths are necessary, they can be produced by applying edge-subdivision, that maintains validity according to Lemma 5.9.5. Again, the new inequality is of type (5.12). □

As the question of validity is now settled for antiweb-s-wheels we turn next to the question of facetness.

Theorem 5.10.7 (proper antiweb-s-wheel facets for $n \equiv t-1 \bmod t$)**.** *Given a proper (n,t)-antiweb-s-wheel G with $n \equiv t-1 \bmod t$ the inequality* (5.12) *induces a facet of $STAB(G)$.*

PROOF. We will first show that the theorem is true for basic (n,t)-antiweb-s-wheels. Given an (n,t)-antiweb-s-wheel $H^{v_{0_1},v_{0_2},\ldots,v_{0_s}}$ star subdivision is applicable to every vertex of H in $H^{v_{0_1},v_{0_2},\ldots,v_{0_s}}$ and double-edge subdivision is applicable to every edge of H in $H^{v_{0_1},v_{0_2},\ldots,v_{0_s}}$ with respect to

$$\left\lfloor\frac{n}{t}\right\rfloor\sum_{i=1}^{s}x_{0_i}+\sum_{i\in H}x_i\leq\left\lfloor\frac{n}{t}\right\rfloor \tag{5.13}$$

by Theorem 5.9.23. Furthermore, all spokes of $H^{v_{0_1},v_{0_2},\ldots,v_{0_s}}$ could be doubly subdivided.

By Theorem 5.9.15 we can carry out the star subdivision at all vertices of $\mathcal{E}$ in $H^{v_{0_1},v_{0_2},\ldots,v_{0_s}}$ with respect to inequality (5.13). This leads to the desired facet of the basic (n,t)-antiweb-s-wheel H'; the inequality can be written as

$$\left\lfloor\frac{n}{t}\right\rfloor\sum_{i=1}^{s}x_{0_i}+\sum_{i\in\mathcal{O}}x_i+(2t-2)\sum_{i\in\mathcal{E}}x_i+\sum_{v\in\mathcal{S}\cup\mathcal{R}}x_v\leq\left\lfloor\frac{n}{t}\right\rfloor+(2t-2)|\mathcal{E}|.$$

By Theorem 5.9.15 we can finally doubly subdivide the remaining edges (as necessary) and prove thereby the desired facet in general. □

Theorem 5.10.8 (proper antiweb-s-wheel facets $(1 \leq (n \bmod t) \leq t-2)$). *Given a proper (n,t)-antiweb-1-wheel G with $n \equiv a \bmod t$, $1 \leq a \leq t-2$. Then inequality* (5.12) *induces a facet of the corresponding stable set polytope if and only if*

1. $\mathcal{E} = \emptyset$ *and*
2. *all paths $P_{i,j}$ with $\{i,j\}$ of type $> a$ and $i,j \in \mathcal{O}$ have length* 1.

Proof. The proof that 1 and 2 imply facetness is the same as that in Theorem 5.10.7, except that no star subdivision is done, and only edges of type $\leq n \bmod t$ are doubly subdivided (as necessary), while edges of type $> n \bmod t$ are not doubly subdivided (as they do not fulfill the prerequisites of double edge-subdivision).

For the forward direction consider an arbitrary but proper (n,t)-antiweb-s-wheel G' with partition $\mathcal{E}\dot{\cup}\mathcal{O}$ that is facet-inducing for $\mathrm{STAB}(G')$. First we want to apply Proposition 5.9.6 to shorten paths of length 3 to edges while maintaining facetness in the following way:

- paths $P_{i,j}$ of type $> a$ and of length > 3 are shortened until their length is 2 or 3;
- paths $P_{i,j}$ of type $< a$ with both ends in $\mathcal{E}$ are shortened down to length 3;
- paths $P_{i,j}$ of type $< a$ with at most one end in $\mathcal{E}$ are shortened down to length 1 or 2;
- spoke-paths are shortened down to length 1 or 2.

Denote with G the resulting graph. Now if G' violates condition 1 or 2 then G does, as we did not change $\mathcal{E}$ at all and even though we shortened the subdivided paths, we took care that paths violating condition 2 are not changed in any way.

Now consider a sequence of undoing star subdivisions and shortening paths of length 3 to edges to reduce G to an unsubdivided (n,t)-antiweb-wheel H. Notice, that by Lemma 5.9.11 the inequality after undoing a single star subdivision is again facet inducing. By Proposition 5.9.6 follows similarly that the inequality after shortening a path is again facet inducing.

Now consider the last intermediate graph in this sequence H'. By Theorem 5.9.23 we know that neither star subdivision nor double-edge subdivision are applicable to H'. If the last operation is an star subdivision, then by Lemma 5.9.10 it follows that going from H to H' by star subdivision destroys facetness that is, the inequality for H' is not facet inducing contrary to assumption. If on the other hand the last operation is doubly subdividing an edge of type $> a$ then by the remark after Proposition 5.9.4 and the fact that we have a facet for H' we obtain that that edge does not violate condition (2) giving again a contradiction. $\square$

So we have a complete characterization of the facet defining proper antiweb-s-wheels.

5.11. Improper Antiweb-Wheels

The next natural question is: "How about the improper antiweb-wheels?" We would like to know too...

The only difference between proper and improper antiweb-wheels is the requirement that for proper ones every path emanating from a vertex of $\mathcal{E}$ must have length at least two. So every improper antiweb-wheel has a path between two members of $\mathcal{E}$ of length 1 (instead of 3 for proper ones). We have never observed any improper antiweb-1-wheel facet. So we believe that they *are* indeed never facet inducing.

As a stepping-stone for this puzzle we give the next lemma.

Lemma 5.11.1 (the induced C_5)**.**
Consider the stable set problem on a graph $G(V, E)$ and a facet-inducing inequality $a^T x \leq \alpha$. Let $G' = (V', E')$ be the support graph of $a^T x \leq \alpha$ in G. Then G' does not contain an induced 5-cycle $C = \{v_1, v_2, v_3, v_4, v_5\}$ where vertices v_2, v_4 have degree 2 in G' and for the vertex v_1 holds that its weight (with respect to a) is higher than the sum of the weights of its neighbors not in C.

PROOF. We prove the theorem, by showing, that if such a 5-cycle exists then the face $a^T x = \alpha$ is contained in the face induced by the 5-cycle inequality. So consider a stable set S with $a^T x^S = \alpha$. Denote with b the characteristic vector of the odd 5-cycle C. We want to prove that $b^T x^S = 2$ for all stable sets with $a^T x^S = \alpha$. Suppose $a^T x^S = \alpha$ and $b^T x^S = 0$, then it is easy to see, that the set $S' = S \cup \{v_2, v_4\}$ fulfills $a^T x^S > \alpha$ contradicting the validity of $a^T x \leq \alpha$ for all stable sets.

Suppose next $a^T x^S = \alpha$ and $b^T x^S = 1$. This requires $|C \cap S| = 1$. Without loss of generality we can assume $C \cap S \subset \{v_1, v_2, v_3\}$. If $C \cap S \subset \{v_1, v_2\}$ then $S' = S \cup \{v_4\}$ violates $a^T x \leq \alpha$. So it remains to study

$C \cap S = \{v_3\}$. Either $S \cup \{v_1\}$ or $S \cup \{v_5\}$ is stable (and then violates $a^T x \leq \alpha$) or both are not stable, because S contains neighbors of v_1 and v_5 outside of C. Now consider the set $S' = (S \setminus N(v_1)) \cup \{v_1\}$. Again, S' is stable. But, as the sum of the weights of the neighbors of v_1 outside of C is smaller than a_{v_1} we learn $a^T x^{S'} > a^T x^S = \alpha$, contradicting the validity of $a^T x \leq \alpha$. □

This theorem helps to weed out many of the antiweb-wheels that are not facet inducing as demonstrated by the following corollary.

Corollary 5.11.2.
Let G be an improper antiweb-s-wheel ($s \geq 1$) with partition $\mathcal{E}, \mathcal{O}$ and corresponding antiweb-1-wheel inequality I_G. Let $H = G[\mathcal{E}]$. If H contains a vertex of degree 1 then I_G is not facet inducing.

Proof. Consider a vertex $u \in \mathcal{E}$ of degree 1 and its unique neighbor $w \in \mathcal{E}$ and assume that the I_G nevertheless induces a facet. Without loss of generality we can assume that the spoke-paths from u, w to the hub have length two (otherwise we could shorten them with Proposition 5.9.6). Consider the C_5 induced by u, w the edge between them and their two spoke paths. Now notice that u has weight $2t - 3 + s$ and it has only $(2t - 2 + s) - 2$ neighbors outside of this C_5; all of these $(2t - 2 + s) - 2$ neighbors have weight 1. So we might conclude that by the Lemma 5.11.1 the valid inequality does not define a facet, contrary to assumption. □

5.12. Generalizations and Remarks

Using well-known transformations, one gets the valid inequalities corresponding to the new inequalities of this chapter for the cut polytope, as performed for the wheel inequalities for example by Cheng [Che98]; from the cut polytope they carry over to the boolean quadric polytope as demonstrated by De Simone [Sim90]. Similarly, they could be utilized for the multiwaycut problem [BTV95]. But, in fact, the connection between stable set, max-cut, and boolean quadric problems is deeper and holds not only for the polytopes but also for their semi-definite relaxations, see Laurent, Poljak, and Rendl [LPR97].

CHAPTER 6

Partial Substitution for Stable Set Polytopes

We introduce the new graph operation of *partial substitution* which generalizes Chvátal's notion of *substitution* of a graph. Using partial substitution we show how to derive new facets of the stable set polytope of the graph from known facets of some of its subgraphs. These facets were not previously known. Furthermore, we show for one example class of facets that a superclass of these facets can be separated in polynomial time. We also discuss the relation between partial substitution and a composition defined by Cunningham.

6.1. Introduction

In this chapter the facial consequences of composing two graphs together with (some) known facets are studied. The classical result in this direction is by Chvátal [Chv75], that if a full linear description for the stable set polytopes of two graph G_1, G_2 is known then a (possibly non-minimal) linear description of the stable set polytope of G_2 substituted into a vertex of G_1 can be easily constructed.

As an application of some interesting extended formulations in [BW97] Borndörfer and Weismantel (see also [Bor97, Chap. 2]) give the new class of odd cycle of odd cycles inequalities. This class composes the pattern 'odd cycle' with itself; they prove its validity by 'aggregation' and its facetness by a custom-made procedure. The odd cycle of odd cycles is not a substitution in the sense of Chvátal, but these operations are very similar. This motivated us to introduce and study the *partial substitution* for graphs. Partial substitution is a generalization of Chvátal's [Chv75] substitution and is powerful enough to explain facetness for odd cycle of odd cycles and even some further generalizations, see Theorem 6.3.11. It is similar to the composition studied by Cunningham in [CUN82]. But our results are complementary to Cunningham's in that while he was able to describe a set of faces which contained all facets, it remained unknown which of these faces were the facets; by contrast, all faces we construct are proven facets, but we do not know how to generate all facets.

6.2. Preliminaries

As STAB(G) is down monotone (that is $x \in$ STAB(G) and $\mathbf{0} \leq y \leq x$ implies $y \in$ STAB(G)), we know that every of its facets $a^T x \leq f$, where $a \leq \mathbf{0}$, is of the form $-x_i \leq 0$. Therefore, the interesting (non trivial) facets all have $a \gneqq \mathbf{0}$. They will be the main subject of our study.

Next we introduce a new graph-theoretic operation, the *partial substitution.* The partial substitution was devised as a generalization of Chvátal's graph substitution [CHV75]. The relation between composition [CUN82] and partial substitution is further discussed in Section 6.4.

Definition 6.2.1 (Partial Substitution).
Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be graphs with $V_1 \cap V_2 = \emptyset$. Choose a subset L of V_2 and a vertex v of V_1. The partial substitution of G_2 with L into G_1 at v (partial substitution of (G_2, L) in (G_1, v), for short), denoted by $(G_1, v) \lhd (G_2, L)$, is the graph $G = (V, E)$ with $V = V_1 \setminus \{v\} \cup V_2$ and

$$E = E_2 \cup \{e \in E_1 \colon v \notin e\} \cup \{\{u, w\} \colon \{u, v\} \in E_1 \text{ and } w \in L\}\,.$$

For example, if $L = V_2$ we get the familiar graph substitution of [CHV75]; his theorem about the faces of the stable set polytope reads (slightly changed):

Theorem 6.2.2 (Chvátal [CHV75], Thm. 5.1).
Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be graphs with $V_1 \cap V_2 = \emptyset$. For $k \in \{1, 2\}$, let

$$\begin{array}{rcll} -x_u & \leq & 0 & (u \in V_k) \\ \displaystyle\sum_{u \in V_k} a_{iu} x_u & \leq & f_i & (i \in J_k) \end{array}$$

be a defining linear system of STAB(G_k) where J_k and a_i are defined suitable. Scale the inequalities with $j \in J_2$ such that $f_j = 1$. Let v be a vertex of G_1 and let $G = (G_1, v) \lhd (G_2, V_2)$. For each $i \in J_1$, set

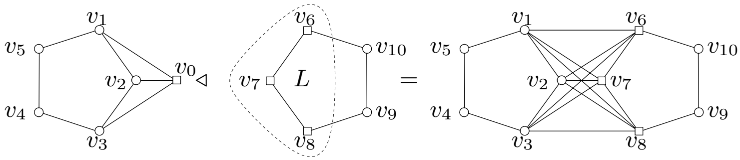


Figure 6.1. The graphs (G_1, v_0) and (G_2, L) and the partial substitution $(G_1, v) \triangleleft (G_2, L)$.

$a_{iv}^+ = \max\{a_{iv}, 0\}$. *Then*

$$
\begin{array}{rrcll}
 & -\,x_u & \leq & 0 & (u \in V_2 \cup (V_1 \setminus \{v\})) \\
(6.1) & a_{iv}^+ \sum\limits_{u \in V_2} a_{ju} x_u + \sum\limits_{u \in V_1 \setminus \{v\}} a_{iu} x_u & \leq & f_i & (i \in J_1, j \in J_2)
\end{array}
$$

is a defining linear system of STAB(G).

For a 'real' application of partial substitution we give the next example:

Example 6.2.3.
Consider the two graphs G_1 and G_2 ($= C_5$) in the left part of Figure 6.1. Let $V_1 = \{v_1, v_2, v_3, v_4, v_5, v_0\}$, $E_1 = \{\{v_1, v_2\}, \{v_2, v_3\}, \{v_3, v_4\}, \{v_4, v_5\}, \{v_5, v_1\}\} \cup \{\{v_i, v_0\} \colon 1 \leq i \leq 3\}$, $V_2 = \{v_6, v_7, v_8, v_9, v_{10}\}$, and $E_2 = \{\{v_6, v_7\}, \{v_7, v_8\}, \{v_8, v_9\}, \{v_9, v_{10}\}, \{v_{10}, v_6\}\}$. Let $L = \{v_6, v_7, v_8\}$. The graph obtained when L of G_2 is substituted for v_0 in G_1, is depicted in the right part of Figure 6.1. Notice that this graph cannot be obtained by substitution of smaller graphs.

The goal of this section is to construct new facets of STAB$((G_1, v) \triangleleft (G_2, L))$ from the knowledge of facets of STAB(G_1) and STAB(G_2).

6.3. Partial Substitution for Facets of Stable Set Polyhedra

In this section we will study different types of facets of STAB$((G_1, v) \triangleleft (G_2, L))$. In the following, we always assume that nontrivial facet defining inequalities $a_1^T z_1 \leq f_1$ and $a_2^T z_2 \leq f_2$ are given for STAB(G_1) and STAB(G_2), respectively.

A set $L \subset V_2$ will be called an *essential set* for (a_2, f_2) if for all stable sets $U \subseteq (V_2 \setminus L)$ the inequality

$$a_2^T z_2^U \leq f_2 - 1 \tag{6.2}$$

is valid (z_2^U denotes the characteristic vector of the set U) and such that there is a stable set W with $a_2^T z_2^W = f_2$ and $a_2^T z_2^{W \cap L} = 1$ (this might require proper scaling of (a_2, f)). Define $b_2 = z_2^W$. Hence for all stable sets U of G_2 with $a_2^T z_2^U = f_2$ follows $L(z_2^U) \geq 1$, where $L(z_2)$ is defined by $L(z_2) = \sum_{i \in L} a_{2i} z_{2i}$.

Now we are prepared to give the main class of new facets.

Theorem 6.3.1 (Facets of Type 1).
Let $\sum_{u \in V_1} a_{1u} x_u \leq f_1$ be a facet defining inequality of STAB(G_1) with $a_{1v} \neq 0$; let $\sum_{u \in V_2} a_{2u} x_u \leq f_2$ with $f_2 > 0$ be a facet defining inequality of STAB(G_2) (properly scaled) such that L is an essential set for (a_2, f_2). Assume that both are nontrivial ($a_1, a_2 \geq 0$). Then

$$a_{1v} \sum_{w \in V_2} a_{2w} x_w + \sum_{w \in V_1 \setminus \{v\}} a_{1w} x_w \leq a_{1v}\,(f_2 - 1)\; + f_1 \tag{6.3}$$

defines a facet of STAB$((G_1, v) \lhd (G_2, L))$.

PROOF. Let a denote the coefficient vector of the left hand side of inequality (6.3), while f denotes the right hand side. Notice $a_w = a_{1w}$ for $w \in V_1 \setminus \{v\}$ and $a_w = a_{1v} a_{2w}$ for $w \in V_2$. The validity of this inequality is a consequence of the aggregation technique of Borndörfer and Weismantel [BW97] (see also [BOR97, Chap. 2]). For the sake of completeness we give a direct proof here. As G_2 is an induced subgraph of $G = (G_1, v) \lhd (G_2, L)$ the inequality

$$\sum_{w \in V_2} a_{2w} x_w \leq f_2 \tag{6.4}$$

is valid for STAB$\,(G)$ and by the same reason

$$\sum_{w \in V_1 \setminus \{v\}} a_{1w} x_w \leq f_1 \tag{6.5}$$

is valid for STAB$\,(G_1 - v)$. Now we have to show that inequality (6.3) is valid for all incidence vectors z^U of stable sets U of G. We have to distinguish two cases depending on $U \cap L$. If $U \cap L = \emptyset$ then $L(x^U) = 0$ and therefore $a_{1v} \sum_{w \in V_2} a_{2w} z_w^U \leq a_{1v}(f_2 - 1)$. Adding this inequality to (6.5) proves $a_{1v}\sum_{w\in V_2} a_{2w}x_w^U + \sum_{w\in V_1\setminus\{v\}} a_{1w}x_w^U \le a_{1v}(f_2-1) + f_1$. But if $U\cap L \neq \emptyset$ then the set $(U\cap V_1)\cup\{v\}$ is stable in G_1. Using (6.5): $a_1^T z_1^{(U\cap V_1)\cup\{v\}} \le f_1$ and $a_1^T z_1^{(U\cap V_1)\cup\{v\}} = a_1^T z_1^{(U\cap V_1)} + a_{1v}$ we obtain the following strengthening: $\sum_{w\in V_1\setminus\{v\}} a_{1w}z_{1w}^{U\cap V_1} \le f_1 - a_{1v}$. Adding this to (6.4) yields $a_{1v}\sum_{w\in V_2} a_{2w}x_w^U + \sum_{w\in V_1\setminus\{v\}} a_{1w}x_w^U \le a_{1v}(f_2-1) + f_1$.

For the proof of facetness we will assume, that there is another valid and facet-defining inequality $c^T x \le c_0$ which defines a facet of STAB(G) containing the face F defined by (6.3) and then we will show that $c^T x \le c_0$ can only be a positive multiple of $a^T x \le f$. Denote with F_1 and F_2 the faces induced by $\sum_{u\in V_1} a_{1u}x_u \le f_1$ and $\sum_{u\in V_2} a_{2u}x_u \le f_2$ of STAB(G_1) and STAB(G_2), respectively.

In a first step in the proof of facetness we will show that $c_j = \text{const}\cdot a_j$ for all $j\in V_2$. In the second step we will extend this result to all $j\in V$.

Let B_1 be a matrix whose column vectors are vertices of F_1 and minimally span F_1. The matrix B_2 is defined analogously. For $z_1\in B_1$ (here we consider B_1 as a set of column vectors) and $z_2\in B_2$ we define $(z_1^T, z_2^T)^T$ as the concatenation of the two vectors, where the component of z_1 with index v is dropped. Let b_1 be a vertex of F_1 with $b_{1v}=1$.

Notice, that $(z_1,z_2)\in F\cap\{0,1\}^{|V_1|+|V_2|-1}$ (for $z_1\in B_1$ and z_2 is the incidence vector of a stable set in G_2) is equivalent either to $z_{1v}=1$ (and $z_2\in F_2$ arbitrary) or to $z_{1v}=0$, $L(z_2)=0$ and $a_2^T z_2 = f_2-1$.

So choose $j,j'\in V_2$ such that $a_{2j}\neq 0$. Notice, that $\frac{f_2}{a_{2j}}e_j \in \operatorname{aff} F_2 = \operatorname{aff} B_2$. Hence there exists a real vector λ, indexed by elements of B_2, with $\sum_{z_2\in B_2}\lambda_{z_2}z_2 = \frac{f_2}{a_{2j}}e_j$ and $\sum_{z_2\in B_2}\lambda_{z_2}=1$. Recall that $(b_1^T, z_2^T)^T\in F$ for all $z_2\in B_2$. Now let $x=\sum_{z_2\in B_2}\lambda_{z_2}(b_1^T,z_2^T)^T\in F$.

Similar, if $a_{2j'}\neq 0$ then there exists λ' with $\sum_{z_2\in B_2}\lambda'_{z_2}z_2 = \frac{f_2}{a_{2j'}}e_{j'}$ and $\sum_{z_2\in B_2}\lambda'_{z_2}=1$. Let $x'=\sum_{z_2\in B_2}\lambda'_{z_2}(b_1^T,z_2^T)^T\in F$.

As $x,x'\in F$ we know that $c^T x = c_0$ and $c^T x' = c_0$ and thereby $c^T(x-x')=0$. But notice that $x-x' = \frac{f_2}{a_{2j}}e_j - \frac{f_2}{a_{2j'}}e_{j'}$. Hence we obtain $0 = c^T(x-x') = c_j\frac{f_2}{a_{2j}} - c_{j'}\frac{f_2}{a_{2j'}}$ and finally $c_{j'} = \frac{c_j}{a_{2j}}a_{2j'}$.

If on the other hand $a_{2j'}=0$ then $\frac{f_2}{a_{2j}}e_j + e_{j'}\in\operatorname{aff} F_2$ and there exists λ' with $\sum_{z_2\in B_2}\lambda'_{z_2}z_2 = \frac{f_2}{a_{2j'}}e_{j'} + e_{j'}$ and $\sum_{z_2\in B_2}\lambda'_{z_2} = 1$. Let $x'=\sum_{z_2\in B_2}\lambda'_{z_2}(b_1^T,z_2^T)^T\in F$.

As $x,x'\in F$ we know that $c^T x = c_0$ and $c^T x' = c_0$ and thereby $c^T(x-x')=0$. But notice that $x-x'=e_{j'}$. Hence we obtain $c_{j'}=0$ and thereby $c_{j'} = \frac{c_j}{a_{2j}} a_{2j'}$. So we have shown for the first step $c_{j'} = \frac{c_j}{a_{2j}} a_{2j'}$ and hence

$$c_{j'} = \frac{c_j}{a_j} a_{j'}.$$

For the second—and last—step let b_2 be the incidence vector of a stable set W with $a_2^T z_2^W = f_2$ and $a_2^T z_2^{W \cap L} = 1$. Let $c_v = \sum_{w \in L} b_{2w} c_w$. We want to show that all c_i for $i \in V_1 \setminus \{v\}$ are equal to $\frac{c_v}{a_{1v}} a_i$. (Of course, by the first step we have for a $j \in V_2$ with $a_j \neq 0$ that $\frac{c_v}{a_{1v}} = \frac{c_j}{a_j}$.)

For $z_1 \in B_1$ we define $\Pi(z_1, b_2)$ by

$$(\Pi(z_1, b_2))_i = \begin{cases} 0 & : \text{if } i \in L \text{ and } z_{1v} = 0, \\ b_{2i} & : \text{if } i \in L \text{ and } z_{1v} = 1, \text{ and} \\ z_i & : \text{if } i \in V_1 \setminus \{v\}. \end{cases}$$

Notice that for all $z_1 \in B_1$ follows $\Pi(z_1, b_2) \in F$.

Since $a_{1v} > 0$ the vector $\frac{f_1}{a_{1v}} e_v$ belongs to F_1. Hence there exists an affine combination λ with $\sum_{z_1 \in B_1} \lambda_{z_1} z_1 = \frac{f_1}{a_{1v}} e_v$. Let $x = \sum_{z_1 \in B_1} \lambda_{z_1} \Pi(z_1, b_2)$. Choose $j' \in V_1 \setminus \{v\}$. Again, there exists an affine combination λ' with $\sum_{z_1 \in B_1} \lambda'_{z_1} z_1$ either equal to $\frac{f_1}{a_{1j'}} e_{j'}$ (if $a_{1j'} \neq 0$) or equal to $\frac{f_1}{a_{1v}} e_v + e_{j'}$ (if $a_{1j'} = 0$); let $x' = \sum_{z_1 \in B_1} \lambda'_{z_1} \Pi(z_1, b_2)$. Now we can exploit $c^T x - c^T x' = 0$ to obtain $c_{j'} = \frac{c_v}{a_{1v}} a_{1j'}$.

With the two steps we have succeeded now in proving for any fixed $j \in V_2$ with $a_j \neq 0$ (and such j exists, as $L(b_2) \neq 0$) that $(c, c_0) = \frac{c_j}{a_j}(a, f)$. As claimed $\frac{c_j}{a_j}$ is nonnegative, since otherwise the right hand side of the facet defining inequality $c^T x \leq f$ would be strictly negative which is impossible. □

The next theorem shows that facets of STAB(G_1) with $a_{1v} = 0$ carry over to STAB($(G_1, v) \triangleleft (G_2, L)$).

Theorem 6.3.2 (Facets of Type 2).
Let $\sum_{u \in V_1} a_{1u} x_u \leq f_1$ be a facet defining inequality of STAB(G_1) with $a_{1v} = 0$. Let $\sum_{u \in V_2} a_{2u} x_u \leq f_2$ be an arbitrary valid inequality of STAB(G_2). Then

$$a_{1v} \sum_{w \in V_2} a_{2w} x_w + \sum_{w \in V_1 \setminus \{v\}} a_{1w} x_w \leq a_{1v} (f_2 - 1) + f_1 \tag{6.6}$$

defines a facet of STAB($(G_1, v) \triangleleft (G_2, L)$).

Remark 6.3.3.
Notice that due to $a_{1v} = 0$ the inequality (6.6) actually reads $\sum_{u \in V_1} a_{1u} x_u \leq f_1$ and the particular choice of (a_2, f_2) does not matter. But this notation permits a simple way to state Theorem 6.3.5 later.

Proof of Theorem 6.3.2. As $G_1 - \{v\}$ (which contains the support of a_1) is an induced subgraph of $(G_1, v) \triangleleft (G_2, L)$ validity follows immediately. Since inequality (6.6) induces a facet of $\mathrm{STAB}(G_1)$, there exists a full-rank matrix B of those column vectors which belong to $\mathrm{STAB}(G_1)$ and fulfill (6.6) with equality. Furthermore, B contains a column b with $b_v = 0$ and another one c with $c_v = 1$. Assume, without loss of generality that the last row of B corresponds to v. Let b' and c' be the vectors b and c, respectively, where the last row is deleted. Notice that for $u \in V_2 \setminus L$ the vectors $b' + e_u$ correspond to stable sets of $(G_1, v) \triangleleft (G_2, L)$. The same holds for $c' + e_w$ for all $w \in L$. Furthermore, these $b' + e_u$ and $c' + e_w$ fulfill (6.6) with equality, as do all columns of B (expanded with some zeroes). Now we can write down all these vectors (assuming that the first rows correspond to $V_1 \setminus \{v\}$ the next to L and the last ones to $V_2 \setminus L$):

$$B^{\text{new}} = \left(\begin{array}{c|c|c} B & c' \cdots c' & b' \cdots b' \\ \cline{2-3} & 0 \cdots 0 & \\ \cline{2-2} \mathbf{0}_{|L|-1,|V_1|} & \mathbf{I}_{|L|-1,|L|-1} & \mathbf{0}_{|L|,|V_2|-|L|} \\ \hline \mathbf{0}_{|V_2|-|L|,|V_1|} & \mathbf{0}_{|V_2|-|L|,|L|-1} & \mathbf{I}_{|V_2|-|L|,|V_2|-|L|} \end{array}\right)$$

As B has full rank and the remaining matrix has only nonzeros on and above the diagonal, and on the diagonal are only 1's we can conclude that B^{new} has full rank and hence the inequality (6.6) induces indeed a facet of $\mathrm{STAB}((G_1, v) \triangleleft (G_2, L))$. □

Next we study the case of an inequality $\sum_{w \in V_2} a_{2w} x_w \leq f_2$ for which L is never essential.

Theorem 6.3.4 (Facets of Type 3)**.**
Let $\sum_{w \in V_2} a_{2w} x_w \leq f_2$ be a facet defining inequality of $STAB(G_2)$ and there is no scaling of a_2 such that L is an essential set for a_2. (Hence there

exists a stable set $U \subseteq V_2 \setminus L$ *with* $\sum_{w \in V_2} a_{2u} x_w^U = f_2$*.) Assume that the inequality is nontrivial (that is* $a_2 \geq 0$*). Then*

$$\sum_{w \in V_2} a_{2w} x_w \leq f_2 \tag{6.7}$$

defines a facet of $STAB((G_1, v) \lhd (G_2, L))$.

PROOF. As G_2 is an induced subgraph of $(G_1, v) \lhd (G_2, L)$ validity follows directly. As inequality (6.7) induces a facet of STAB(G_2), there exists a full-rank matrix B of those column vectors which belong to STAB(G_2) and fulfill (6.7) with equality. Furthermore, as L is not essential for any scaling of a_j, there exists a stable set $U \subseteq V_2 \setminus L$ of G_2 with $\sum_{w \in V_2} a_{2w} x_w^U = f_2$. Let $b_2 = x^U$. Observe, that $U \cup \{u\}$ for any $u \in V_1 \setminus \{v\}$ is a stable set of $(G_1, v) \lhd (G_2, L)$ and the corresponding incidence vector fulfills (6.7) with equality. Therefore, all column vectors of the matrix

$$B' = \left(\begin{array}{c|c} \mathbf{0}_{|V_1|-1, |V_2|} & \mathbf{I}_{|V_1|-1, |V_1|-1} \\ \hline B & b_2 \cdots b_2 \end{array} \right)$$

correspond to stable sets of $(G_1, v) \lhd (G_2, L)$. Notice that, as B' has full rank, the inequality (6.7) defines a facet of STAB($(G_1, v) \lhd (G_2, L)$). □

The previous three theorems can immediately be used to strengthen the theorem of Chvátal in the following way:

Theorem 6.3.5.
Consider the same assumptions as in Theorem 6.2.2 with the additional requirement that all given faces are facets. It follows that all constructed faces are facets.

For a graph G and a set $L \subseteq V(G)$ we denote with $G[L] = (V', E')$ the subgraph of G induced by L, where $V' = L$ and $E' = \{\{u, v\} \in E \colon u, v \in L\}$.

Theorem 6.3.6 (Facets of Type 4)**.**
Let $\sum_{w \in V_1} a_{1w} x_w \leq f_i$ *be a facet defining, nontrivial inequality of* $STAB(G_1)$ *and* $\sum_{w \in L} a_{2w} x_w \leq 1$ *be a facet defining, nontrivial inequality*

of STAB($G_2[L]$). Then

$$a_{iv} \sum_{u \in L} a_{ju} x_u + \sum_{u \in V_1 \setminus \{v\}} a_{iu} x_u \leq f_i \tag{6.8}$$

defines a facet of STAB($(G_1, v) \lhd (G_2, L)$).

PROOF. Notice that L is essential for a_2. With Theorem 6.3.1 follows immediately, that (6.8) is valid and facet defining for STAB($(G_1, v) \lhd (G_2[L], L)$). As $(G_1, v) \lhd (G_2[L], L)$ is an induced subgraph of $(G_1, v) \lhd (G_2, L)$ validity of (6.8) for STAB($(G_1, v) \lhd (G_2, L)$) is a direct consequence. Let B be a quadratic matrix of those zero-one column-vectors, which are in STAB($(G_1, v) \lhd (G_2[L], L)$) and fulfill (6.8) with equality, such that B has full rank. Let $w \in V_1$ be a neighbor of v in G_1. There must be a vector b in B with $b_w = 1$, otherwise B would be contained in the plane $x_w = 1$. Notice that, as $b_w = 1$, b is the incidence vector of a stable set, and all elements $u \in L$ are neighbors of w in $(G_1, v) \lhd (G_2[L], L)$ and therefore for all $u \in L$ holds $b_u = 0$.

Let U be the stable set corresponding to b. Notice that for all $t \in V_2 \setminus L$ the set $U \cup \{t\}$ is stable in $(G_1, v) \lhd (G_2, L)$ and the incidence vectors fulfill (6.8) with equality. Hence the columns of the following matrix B' belong to STAB($(G_1, v) \lhd (G_2, L)$) and fulfill (6.8) with equality.

$$B' = \left(\begin{array}{c|c} B & b \cdots b \\ \hline \mathbf{0}_{|V_2|-|L|,|V_1|} & \mathbf{I}_{|V_2|-|L|,|V_2|-|L|} \end{array} \right)$$

Notice that B' has full rank. □

In this section we described four classes of facets for the stable set polytope of a graph $(G_1, v) \lhd (G_2, L)$. In particular, the first three classes of facets generalize the facets of Chvátal [CHV75, Theorem 5.1].

The next natural question is, whether these four classes constitute already a defining system for STAB($(G_1, v) \lhd (G_2, L)$). Unfortunately, this is not true as we demonstrate in the following remark.

Remark 6.3.7.
*Observe, that there are graphs G, sets L, and valid (not facet-defining) inequalities $a^T x \leq f$ for $STAB(G)$ which are **not** implied by other facet-defining, essential inequalities. Consider for example the graph*

$$P_4 = G\left(\{1,2,3,4\}, \{\{1,2\},\{2,3\},\{3,4\}\}\right)$$

and $L = \{1,4\}$. Notice that the inequality $x_1 + x_2 + x_3 + x_4 \leq 2$ is valid but not facet-defining for $STAB(P_4)$ and that L is essential for this inequality. The only nontrivial facets of $STAB(P_4)$ are

$$\begin{aligned} x_1 + x_2 &\leq 1 \\ x_2 + x_3 &\leq 1 \\ x_3 + x_4 &\leq 1. \end{aligned}$$

However, L is not essential for any of these four facets.

For example, if we partially substitute P_4 with $\{1,4\}$ into a K_2 then the resulting graph is a C_5 (for which we know that the rank inequality defines a facet) while the rank inequality is not of any of the previously described types. So there are still more facets.

Finally, we show how to apply Theorem 6.3.1 iteratively so as to prove facetness of a generalization of the odd cycle of odd cycles inequalities of Borndörfer and Weismantel [BW97]. Additionally, we show that a superclass of these inequalities can be separated in polynomial time.

Theorem 6.3.8 (Iterative application of Theorems 6.3.1, 6.3.2)**.**
Let $\sum_{u \in V_0} a_{0u} x_u \leq f_0$ be a facet defining inequality of $STAB(G_0)$; fix a number m and a set $\{v_1, v_2, \ldots, v_m\}$ of vertices of G_0. Let m additional graphs $G_1, G_2, \ldots, G_m$ be given, which have mutually disjoint vertex sets and choose for all $i \in \{1,2,\ldots,m\}$ sets $L_i \in V_i$. For $i \in \{1,2,\ldots,m\}$ choose facet defining inequalities $\sum_{u \in V_i} a_{iu} x_u \leq f_i$ of $STAB(G_i)$ such that L_i is essential. Finally, assume that all inequalities are nontrivial. Then

$$\text{(6.9)} \qquad \sum_{i=1}^{m} a_{0v_i} \sum_{w \in V_i} a_{iw} x_w + \sum_{w \in V_0 \setminus \{v_1, v_2, \ldots, v_m\}} a_{0w} x_w \leq \sum_{i=1}^{m} a_{0v_i} (f_i - 1) \; + f_0$$

defines a facet of $STAB((\cdots(((G_0, v_1) \lhd (G_1, L_1), v_2) \lhd (G_2, L_2), v_3) \cdots \lhd (G_m, L_m)))$.

PROOF. The proof is by inductively showing, that (for $m' = 1, 2, \ldots, m$)

$$\sum_{i=1}^{m'} a_{0v_i} \sum_{w \in V_i} a_{iw} x_w + \sum_{w \in V_0 \setminus \{v_1, v_2, \ldots, v_{m'}\}} a_{0w} x_w \leq \sum_{i=1}^{m'} a_{0v_i} (f_i - 1) \ + f_0$$

defines a facet of $\mathrm{STAB}((\cdots(((G_0, v_1) \triangleleft (G_1, L_1), v_2) \triangleleft (G_2, L_2), v_3) \cdots \triangleleft (G_{m'}, L_{m'}))$. Notice that the case $m' = 1$ is a direct application of Theorem 6.3.1 (if $a_{0v_{m'}} \neq 0$) or of Theorem 6.3.2 (if $a_{0v_{m'}} = 0$). The induction step is also a direct application of Theorem 6.3.1 or of Theorem 6.3.2 (depending on $a_{0v_{m'+1}}$). □

As a shortcut in the case of $m = |V_0|$ we write $G_0((G_1, L_1), (G_2, L_2), \ldots, (G_m, L_m)) = (\cdots(((G_0, v_1) \triangleleft (G_1, L_1), v_2) \triangleleft (G_2, L_2), v_3) \cdots \triangleleft (G_m, L_m)$. For simplicity of notation, we will consider the graph $C_0 = (\{1\}, \emptyset)$ an odd hole like the other odd holes C_{2k+1} for $k \geq 1$. For C_0 the odd hole constraint $\mathcal{O}_0$ is $x_1 \leq 1$ while for the C_{2k+1} the odd hole constraint $\mathcal{O}_k$ is $\sum_{i=1}^{2k+1} x_i \leq k$. If P is an even path and subgraph of C_{2k+1} then P is essential for $\frac{2}{|P|-1}\mathcal{O}_k$ if $k \geq 1$. If $k = 0$ (and thereby $|P| = 1$) then P is essential for $\mathcal{O}_0$. The independence number $\alpha(G)$ of a graph G is the size of a maximal stable set in G. Notice $\alpha(C_{2k+1}) = k$ for $k \geq 0$ and $\alpha(C_0) = 1$. Thereby, we obtain for odd l and an even subpath P of C_l that P is essential for $\frac{1}{\alpha(P)}\mathcal{O}_{C_l}$.

Definition 6.3.9 (odd hole of odd holes)**.**
A graph G shall be called odd hole of odd holes *if there exists a natural number $m \in \mathbb{N}$, odd holes $G_0, G_1, \ldots, G_m$, where $G_1, G_2, \ldots, G_m$ contain even (length) paths $L_1, L_2, \ldots, L_m$ such that G is isomorphic to $G_0((G_1, L_1), (G_2, L_2), \ldots, (G_m, L_m))$. The odd hole of odd holes G shall be called simple if V_i and V_j are disjoint for $1 \leq i < j \leq m$. The* order *of the odd hole of odd holes shall be the number* $\max_{1 \leq i \leq m} |L_i|$.

Remark 6.3.10.
For an example of a simple odd hole of odd holes (of order 5*) see Figure 6.2. Furthermore, notice that the (less general) cycle of cycles introduced in* [BW97] *are all odd hole of odd holes of order* 3.

Theorem 6.3.11.
For a simple odd hole of odd holes $G = G_0((G_1, L_1), (G_2, L_2), \ldots, (G_m, L_m))$ the inequality

$$\sum_{i=1}^{m} \sum_{w \in V_i} \frac{1}{\alpha(L_i)} x_w \leq \sum_{i=1}^{m} \left(\frac{\alpha(G_i)}{\alpha(L_i)} - 1 \right) + \alpha(G_0) \tag{6.10}$$

is valid and facet defining for STAB(G).

PROOF. The proof is a simple application of Theorem 6.3.8 and the observation, that L_i is essential for $\frac{1}{\alpha(L_i)}$ times the odd-hole constraint of G_i. □

Theorem 6.3.12.
For an odd hole of odd holes $G = G_0((G_1, L_1), (G_2, L_2), \ldots, (G_m, L_m))$ the inequality

$$\sum_{i=1}^{m} \sum_{w \in V_i} \frac{1}{\alpha(L_i)} x_w \leq \sum_{i=1}^{m} \left(\frac{\alpha(G_i)}{\alpha(L_i)} - 1 \right) + \alpha(G_0) \tag{6.11}$$

is valid for STAB(G).

PROOF. Consider first the graph $G' = G_0((G_1', L_1'), (G_2', L_2'), \ldots, (G_m', L_m'))$ where G_i and G_i' are isomorph; they differ only in that the isomorphisms are chosen to ensure that the different G_i are disjoint. For this graph the inequality (6.11) (properly relabeled for the graph G') is valid. Now notice that

$$\text{ESTAB}(G) = \text{ESTAB}(G') \cap \bigcap_{v' \in V_i', w' \in V_j \text{ with } v=w} \{x \in [0,1]^{|V(G')|} \mid x_{v'} = x_{w'}\}.$$

As validity is maintained under intersecting with hyperplanes, the inequality (6.11) is additionally valid for G. □

Similarly, a K_2 of odd holes and a K_2 of odd cycles can be defined.

Definition 6.3.13 (K_2 of odd holes)**.**
A graph G shall be called K_2 of odd holes *if there exists two odd holes G_1, G_2 containing even (length) paths L_1, L_2 such that G is isomorphic to $K_2((G_1, L_1), (G_2, L_2))$. The* order *of the K_2 of odd holes is again the number* $\max_{1 \leq i \leq m} |L_i|$.

The following theorem is proved similarly as Theorems 6.3.11 and 6.3.12.

Theorem 6.3.14.
For K_2 of odd holes $G = G_0((G_1, L_1), (G_2, L_2))$ the inequality

$$\sum_{i=1}^{2} \sum_{w \in V_i} \frac{1}{\alpha(L_i)} x_w \leq \sum_{i=1}^{2} \left(\frac{\alpha(G_i)}{\alpha(L_i)} - 1 \right) + 1 \tag{6.12}$$

is valid for STAB(G). If the configuration is simple the valid inequality is additionally facet inducing.

Theorem 6.3.15 (Polynomial separation).
The separation problem whether a given $x \in ESTAB(G)$ violates any K_2 of odd holes or odd hole of odd holes inequalities of order $\leq k$, where k is an arbitrary constant, can be solved in polynomial time.

PROOF. (The proof is similar to but a lot more general than the proof given in [BW97].) Assume a graph $G(V, E)$ and a (fractional) vector $x^* \in$ ESTAB(G) are given. Consider the set $\mathcal{V}$ of all even paths (= paths of even length) of size at most k ($|\mathcal{V}| = O(|V|^k)$). On the set $\mathcal{V}$ we want to define a graph $\mathcal{G}(\mathcal{V}, \mathcal{E})$. Two vertices L_1, L_2 of $\mathcal{G}$ are adjacent, iff all vertices of L_1 are adjacent to all vertices of L_2 in G, that is $G[L_1 \cup L_2] \supset K_{|L_1|,|L_2|}$. To assign weights w_{L_i} to each vertex L_i of $\mathcal{G}$ we calculate the shortest odd (length) path P in G that connects the two ends of L_i with respect to the length function

$$l_{uv} = \frac{1 - x_u^* - x_v^*}{2}$$

for all adjacent u, v in G. Denote with C_i the odd cycle $P \cup L_i$. Finally, we can define the weight function of $\mathcal{G}$ for $|L_i| \geq 3$ by

$$w_{L_i} = \frac{1}{\alpha(L_i)} \sum_{w \in C_i} x_w^* - \left(\frac{\alpha(C_i)}{\alpha(L_i)} - 1 \right)$$

and for $|L_i| = 1$ by $w_{L_i} = x_{L_i}^*$.

Now we can check whether the edge inequalities in $\mathcal{G}$ are violated. If any violated edge inequality is found, it corresponds to a maximally (with respect to the vertices in $\mathcal{G}$) violated K_2 of odd cycles inequality; this inequality can be returned. But if all edge inequalities are fulfilled in $\mathcal{G}$, a standard algorithm can be used to find an odd cycle C_0 in $\mathcal{G}$ which violates an odd-hole constraint. If such a cycle exists, say

$V(G_0) = \{1, 2, \ldots, m\}$, then the inequality of the odd hole of odd holes $G_0((G_1, L_1), (G_2, L_2), \ldots, (G_m, L_m))$ is violated.

To see, why the inequality of the odd hole of odd holes $G_0((G_1, L_1), (G_2, L_2), \ldots, (G_m, L_m))$ is violated, assume, that the odd cycle constraint of G_0 is violated by ϵ, that is: $\sum_{i=1}^{m} w_{L_i} = \alpha(G_0) + \epsilon$. Now we can expand the left hand side of the last equation to obtain

$$\sum_{i=1}^{m} \left(\frac{1}{\alpha(L_i)} \sum_{w \in G_i} x_w^* - \left(\frac{\alpha(G_i)}{\alpha(L_i)} - 1 \right) \right) = \sum_{i=1}^{m} \frac{1}{\alpha(L_i)} \sum_{w \in G_i} x_w^* - \sum_{i=1}^{m} \left(\frac{\alpha(G_i)}{\alpha(L_i)} - 1 \right).$$

This implies

$$\sum_{i=1}^{m} \frac{1}{\alpha(L_i)} \sum_{w \in G_i} x_w^* = \sum_{i=1}^{m} \left(\frac{\alpha(G_i)}{\alpha(L_i)} - 1 \right) + \alpha(G_0) + \epsilon.$$

Hereby, we have demonstrated that the odd hole of odd holes inequality is indeed violated.

For the other direction, we have to assume that x^* violates a non-simple odd hole of odd hole inequality of order $\leq k$. Then we can study of course a most violated inequality of this type, say $G_0((G_1, L_1), (G_2, L_2), \ldots, (G_m, L_m))$. Assume, that this inequality is violated by ϵ. First it is important to notice that the path joining the ends of L_i in G_i but avoiding internal vertices of L_i has the same length as P, for otherwise that path could be replaced by P thereby giving a more violated inequality. Now we can look at the length of the cycle corresponding to G_0 in $\mathcal{G}$. Reversing the argument of the preceding paragraph demonstrates that the cycle-inequality corresponding to G_0 in $\mathcal{G}$ is violated by ϵ. □

6.4. Relation to Cunningham's Composition

Denote with $N(v)$ the set of neighbors of v in a given graph. Now we want to study the relation of partial substitution to a composition defined by Cunningham [CUN82].

Definition 6.4.1 (Composition).
Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be graphs with $V_1 \cap V_2 = \emptyset$. Choose vertices $v_1 \in V_1$ and $v_2 \in V_2$ with $N(v_1) \neq \emptyset \neq N(v_2)$. The composition of

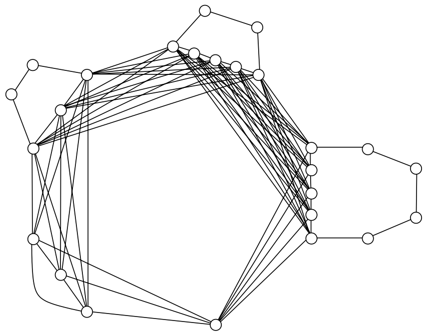


FIGURE 6.2. A simple odd hole of odd holes of order 5.

(G_1, v_1) and (G_2, v_2), denoted by $(G_1, v_1)\triangle(G_2, v_2)$, is the graph $G = (V, E)$ with $V = (V_1 \cup V_2) \setminus \{v_1, v_2\}$ and $E = \{e \in E_1 \colon v_1 \notin e\} \cup \{e \in E_2 \colon v_2 \notin e\} \cup \{\{u, w\} \colon u \in N(v_1)$ and $w \in N(v_2)\}$, where $N(v_1)$ and $N(v_2)$ denote the sets of neighbors of v_1 in G_1, v_2 in G_2, respectively.

For given complete linear descriptions of $\mathrm{STAB}(G_1)$ and $\mathrm{STAB}(G_2)$ a complete linear description of $\mathrm{STAB}((G_1, v_1)\triangle(G_2, v_2))$ is constructed in [CUN82]. But whether this linear complete description is minimal is unknown. Denote with $N_{G_1}(v_1)$ the neighborhood of v_1 in G_1. Now it is an easy observation, that

$$\begin{aligned}(G_1, v_1)\triangle(G_2, v_2) &= (G_1, v_1) \triangleleft (G_2 - v_2, N_{G_2}(v_2)) \\ &= (G_2, v_2) \triangleleft (G_1 - v_1, N_{G_1}(v_1)).\end{aligned}$$

Let $K_2 = (\{1, 2\}, \{\{1, 2\}\})$ then we can state the relation the other way around:

$$(G_1, v) \triangleleft (G_2, L) = (G_1, v)\triangle((K_2, 2) \triangleleft (G_2, L), 1). \tag{6.13}$$

So the question arises, where the polyhedral difference between $\triangle$—where a complete linear description is known, but that description might contain non-facets—and $\triangleleft$—where some facets can be constructed, but not all—stems from. We think the major difficulty lies in Equation (6.13). As demonstrated in Remark 6.3.7 the step in Equation (6.13) where the knowledge of a complete linear description is lost is the step from $\mathrm{STAB}(K_2)$ and $\mathrm{STAB}(G_2)$ to $\mathrm{STAB}(K_2, 2) \triangleleft (G_2, L))$. So going from $\mathrm{STAB}(G_2, v_2)$ to

STAB$(G_2 - v_2)$ is the step, where crucial information is lost. But on the other hand, after this step we are left in circumstances under which we can describe many facets.

All theorems developed here for stable set polyhedra can also be proved in the broader way of independence system polyhedra.

CHAPTER 7

Exact Algorithms for Discrete Tomography

In this chapter we compare two different optimization models for reconstruction problems in discrete tomography. Both models were proposed in the literature and their computational complexities were studied. Though both models have their benefits and applications, they were never practically compared with respect to their utility for solving the reconstruction problem posed by measurements from HRTEM. We compare two different data-scenarios and two solvers. First we will describe the different models; then briefly give the solution programs. For the metric to compare their performance, we do not compare their speed, but more fundamentally we compare how much information can be recovered from the data sets. For this we generate a random configuration (under the chosen model) and next compute another solution (fulfilling all equations) that is as different from the initial configuration as possible. Clearly, those models where this maximal unlikeness (more formally: the maximal symmetric difference) is smaller are (if their assumptions are applicable to the concrete situation) better. Even though for most models configurations do exist so that the reconstruction is unique, we think that by drawing random configurations we get a good empirical picture of their differences. One outcome of these experiments is that one model permits surprisingly different looking reconstructions; this might be very undesirable in some applications.

The results of this chapter are joint work with Peter Gritzmann. Furthermore we thank Jens Zimmermann for helping with coding the datastructures underlying our algorithms.

7.1. A Formulation for Unrestricted Problems

The basic reconstruction problem of discrete tomography as stated in the system of Equations (3.1) (Chapter 3) will be written in this chapter as $Ax = b$ and $x \in \{0,1\}^{n\times n}$. A most different solution to a given solution $\hat{x}$—that is a solution with maximal symmetric difference to the underlying configuration of $\hat{x}$—can be simply described as an optimal solution to the system

$$\max \quad \sum_{i=1}^{n}\sum_{j=1}^{n}(1-\hat{x}_{ij})\cdot x_{ij}$$

subject to

$$Ax = b \tag{7.1a}$$

$$x \in \{0,1\}^{n\times n}. \tag{7.1b}$$

Notice, that if we speak of coordinates (i,j) (as for example in x_{ij}) the first coordinate describes the projection onto the abscissa, the second coordinate describes the projection onto the ordinate. *We do not interpret (i,j) as indices of a matrix, but as coordinates!*

7.2. A Formulation for Line-Convex Problems

Different variants of the unconstrained reconstruction problem are found in the literature. One variant is the reconstruction problem for polyominoes [BLNP96]. For polyominoes the additional requirement is that the set of atoms is connected. Two cells are adjacent, if they are next to each other along a coordinate direction.

Other requirements are about different notions of convexity. For convex lattice sets there are theorems in [GG97] that show that every convex lattice set is uniquely determined by suitable 4 and arbitrary 7 X-rays in pairwise non-parallel coplanar lattice directions. However, it is unknown whether the corresponding reconstruction problem is solvable in polynomial time or not. But it appears, that for our applications of HRTEM to semiconductor-samples requiring convexity of the sample (and the solutions) is too much to ask for. One can think of the objects of interest as wafers only a few hundred atoms thin, which, to begin with, had polished surfaces and then were subject to different etching processes. It is the accuracy of this etching (and related manipulations) that we want to examine. So, even though the object was perhaps convex in the beginning, in the end it is not.

But as we think of it as part of a very large wafer and the etching is performed only from above, it seems reasonable to assume (as a first approximation) that every line that is (almost) perpendicular to the surface of the wafer, intersects the wafer in an interval. If this property holds for all lines of a given direction, the object is called *line-convex* with respect to this direction.

Definition 7.2.1.
A finite set $F \subset \mathbb{Z}^3$ ($\mathbb{Z}^2$) is called line-convex *with respect to a direction l if all translates of l intersect F in an interval. In particular, it is called h-*convex*(v-*convex*) if it is line-convex with respect to* $(1,0,0)$ *(*$(0,1,0)$*).*

Barcucci, Del Lungo, Nivat, and Pinzani [BLNP96] show that the reconstruction problem for h-convex sets with two directions is $\mathbb{NP}$-hard. Similarly, Woeginger [Woe96] shows that the 2-direction reconstruction of sets which are h- and v-convex is $\mathbb{NP}$-hard.

The $\mathbb{NP}$-hard problem most close to the $\mathbb{NP}$-hard/$\mathbb{P}$ border is the problem of h-convex reconstruction for two directions. It is the most simple, yet realistic formulation for our application to problems with convexity along one direction. On the other hand it is the simplest difficult problem. For simplicity of notation, we will study this problem on a 2-dimensional $n \times n$ grid with the directions $(1,0)$ and $(0,1)$.

We will assume, that the data are given as n row sums ($r_j = \sum_{i=1}^n x_{ij}$) and n column sums ($c_i = \sum_{j=1}^n x_{ij}$) where x_{ij} encodes the occupancy status ($= 1$ if there is an atom at position (i,j) and $= 0$ if there is not an atom) of the candidate position (i,j). We say that a vector $(x_i)_{i=1}^n$ has the *strict consecutive ones property* if $\{i\colon x_i = 1\} = \{l, l+1, \ldots, u\}$ for some $l, u \in \mathbb{N}$.

h-convex-Reconstruction.

Instance:	*A number $n \in \mathbb{N}$ and two n-vectors $r, c \in \mathbb{N}^n$.*
Output:	*An element $\bar{x} \in \{0,1\}^{n\times n}$ fulfilling the constraints $r_j = \sum_{i=1}^n x_{ij}$ and $c_i = \sum_{j=1}^n x_{ij}$ so that for every j the vector $(\bar{x}_{ij})_{i=1}^n$ has the strict consecutive ones property.*

An initial approach to formulate this problem as a binary integer program is to write it as a feasibility problem for:

(7.2a) $\qquad Ax = b,$

(7.2b) $(x_{ij})_{i=1}^{n}$ has the strict consecutive ones property for every j, and

(7.2c) $x \in \{0,1\}^{n\times n}$

where A encodes the incidence of the candidate points with the horizontal and vertical lines. However, the h-convexity constraint (7.2b) is not linear (yet).

A first way to obtain a truly integer linear description of h-convex problems is to replace the h-convexity constraint (7.2b) in the preceding problem by constraint (7.3b) in the following system of inequalities

(7.3a) $Ax = b,$

(7.3b) $x_{i_1 j} + x_{i_3 j} - x_{i_2 j} \leq 1$ for all $i_1 < i_2 < i_3$ and all j,

(7.3c) $0 \leq x_{ij} \leq 1$ for $i = 1, 2, \ldots, n$ and for $j = 1, 2, \ldots, n$, and

(7.3d) $x \in \{0,1\}^{n\times n}.$

Denote the convex hull of the solutions of (7.3) by $P_{\mathrm{lc},1}$ and its LP-relaxation (determined by (7.3a), (7.3b), and (7.3c)) $Q_{\mathrm{lc},1}$. The formulation (7.3) gives an integer linear feasibility problem. But it has the major drawback that even for a problem with a single line that is line-convex the corresponding polytope $Q_{\mathrm{lc},1}$ has fractional vertices even though it is of course a trivial task to put the given number of atoms in a line-convex fashion onto that single line. Another disadvantage of this formulation is the very large (though polynomial) number of inequalities. It appears that the main reason for these difficulties is that the encoding of the solution is too sparse. The number of ones in a row of the problem encode the placement of the atoms, but this encoding also permits, in the first place, solutions that are not h-convex. Only later the requirement of h-convexity is added. So it seems more natural to look for a denser encoding, that directly takes care of the h-convexity of the solutions.

Notice, that to describe a solution it suffices to describe the positions where the intervals (in the convex h-direction) start. So we try to use the scheduling variables y_{ij} defined by

$$y_{ij} = \begin{cases} 1 & \text{if the convex interval on the horizontal line } j \\ & \quad \text{starts at position i,} \\ 0 & \text{otherwise.} \end{cases}$$

In other words, $y_{ij} = 1$ means that the atom at position (i, j) and the atoms at the $r_j - 1$ positions to the right of (i, j) are selected. These variables permit a linear way of describing the h-convex-Reconstruction for this problem by the requirements:

$$\sum_{i=1}^{n} y_{ij} = \begin{cases} 0 & \text{if } r_j = 0 \\ 1 & \text{otherwise} \end{cases} \quad \text{for } j = 1, 2, \ldots, n \tag{7.4a}$$

$$\sum_{j=1}^{n} \sum_{k=i-(r_j-1)}^{i} y_{kj} = c_i \text{ for } i = 1, 2, \ldots, n, \tag{7.4b}$$

$$0 \leq y_{ij} \leq 1 \text{ for } i = 1, 2, \ldots, n \text{ and } j = 1, 2, \ldots, n, \tag{7.4c}$$

and

$$y \in \{0, 1\}^{n \times n}. \tag{7.4d}$$

For (i, j) outside the square $[1, n] \times [1, n]$ we set $y_{ij} = 0$. For brevity let $f_i(y) = \sum_{j=1}^{n} \sum_{k=i-(r_j-1)}^{i} y_{kj}$ and denote the convex hull of solutions of (7.4) by $P_{\text{lc},2}$ and its LP-relaxation (determined by (7.4a), (7.4b), and (7.4c)) with $Q_{\text{lc},2}$. A simple observation is that $y_{ij} = 0$ for $j = 1, 2, \ldots, n$ and $i = n - (r_j - 2), \ldots, n$. Therefore, it holds that $\dim P_{\text{lc},2} \leq n^2 - \sum_{j=1}^{n}(r_j - 1)$.

It is natural to ask about the relation between $P_{\text{lc},1}$ and $P_{\text{lc},2}$. It turns out that they are isomorphic under the following linear maps. Define $\Pi_{21} : P_{\text{lc},2} \longmapsto P_{\text{lc},1}$ by

$$(\Pi_{21}(y))_{ij} = \sum_{k=\max(1, i-(r_i-1))}^{i} y_{ij}$$

and $\Pi_{12} : P_{\text{lc},1} \longmapsto P_{\text{lc},2}$ by

$$(\Pi_{21}(x))_{ij} = (x_{ij} - x_{i-1,j}) + \left(x_{i-r_j,j} - x_{i-r_j-1,j}\right) + \ldots .$$

Now it is easy to prove $\Pi_{12}(P_{\text{lc},1}) = P_{\text{lc},2}$ and $\Pi_{21}(P_{\text{lc},2}) = P_{\text{lc},1}$.

Instances with $r_j = r$ for all $j = 1, \ldots, n$ and $c_i = c$ for all $i = 1, \ldots, n$ are called *homogeneous* and instances with $r_j = r$ for all $j = 1, \ldots, n$ are called called *row-homogeneous*. Row-homogeneous instances with the additional property that the sequence of c_i's is not increasing (that is $c_i \geq c_{i+1}$ for $i = 1, 2, \ldots, n - 1$) are called *wedges*.

Theorem 7.2.2.
The polytope $Q_{lc,2}$ is integral for h-convex wedge instances.

PROOF. If $c_1 = 0$ then $c_i = 0$ for all $1 \leq i \leq n$ and the problem has a trivial solution. So we can assume $c_1 > 0$.

We claim that the polytope $Q_{\mathrm{lc},2}$ is contained in the linear subspace

$$\bigcap_{j=1}^{n} \bigcap_{i \not\equiv 1 \bmod r} \{y : y_{ij} = 0\}.$$

Consider an arbitrary solution $\bar{y}$. Notice that $f_i(\bar{y}) \geq f_1(\bar{y})$ for $i = 1, 2, \ldots, r$. But as the sequence of c_i is nonincreasing this implies that $\sum_{i=2}^{r} \sum_{j=1}^{n} \bar{y}_{ij} = 0$. As $\bar{y}$ is nonnegative it follows that $\bar{y}_{ij} = 0$ for $i = 2, 3, \ldots, r$ and $j = 1, 2, \ldots, n$. Now one can continue this proof-step for $i = r+1$ and $i = r+2, r+3, \ldots, 2r$ and so on. So we have shown that

$$Q_{\mathrm{lc},2} \subseteq \bigcap_{j=1}^{n} \bigcap_{i \not\equiv 1 \bmod r} \{y : y_{ij} = 0\}.$$

With these observations we can specialize the description of $Q_{\mathrm{lc},2}$ to

$$\sum_{\substack{1 \leq i \leq n \\ i \equiv 1 \bmod r}} y_{ij} = \begin{cases} 0 & \text{if } r_j = 0 \\ 1 & \text{otherwise} \end{cases} \quad \text{for } j = 1, 2, \ldots, n, \tag{7.5a}$$

$$\sum_{j=1}^{n} y_{ij} = c_i \text{ for } 1 \leq i \leq n \text{ and } i \equiv 1 \bmod r, \tag{7.5b}$$

$$y_{ij} = 0 \text{ for } 1 \leq i \leq n \text{ and } i \not\equiv 1 \bmod r \text{ and}$$

$$0 \leq y_{ij} \leq 1 \text{ for } 1 \leq i \leq n \text{ and } 1 \leq j \leq n. \tag{7.5c}$$

Now notice that the matrix corresponding to the free variables is the node-edge incidence matrix of the $K_{n,n}$, therefore it is totally unimodular (by an easy consequence of the characterization of totally unimodular matrices by Ghouila-Houri [GH62, main theorem]) and the system (7.5) is integral for all instances of wedge-type. (In the special case, that $r = c_i = 1$ for all $1 \leq i \leq n$ the system (7.5) describes an embedding of the Birkhoff polytope into a coordinate plane of a higher-dimensional space. As it is well-known (cf. Birkhoff [BIR46, first theorem] and von Neumann [NEU53, Lemma 2]) that the Birkhoff polytope is integral, we can conclude that $Q_{\mathrm{lc},2}$ is integral.) $\square$

Theorem 7.2.3.
The h-CONVEX-RECONSTRUCTION problem can be solved in polynomial time for h-convex row-homogeneous instances.

PROOF. For this proof we will replace the constraints (7.4b) by simpler but equivalent constraints. Notice that

$$f_i(y) - f_{i-1}(y) = \sum_{j=1}^{n} y_{ij} - \sum_{j=1}^{n} y_{i-r,j}.$$

Let $c'_i = 0$ for $-r+1 \leq i \leq 0$ and $c'_1 = c_1$. Now define successively $c'_i = c_i - c_{i-1} + c'_{i-r}$. Suppose that we have proved $\sum_{j=1}^{n} y_{ij} = c'_i$ for $i \leq k-1$ (and we have done this for $k=2$) then we want it to hold also for k. Then we have by the assumption

$$\begin{aligned} f_k(y) - f_{k-1}(y) + \sum_{j=1}^{n} y_{k-r,j} &= c_k - c_{k-1} + c'_{k-r} \\ &= c'_k. \end{aligned}$$

On the other hand we have by the previous observation that

$$\begin{aligned} f_k(y) - f_{k-1}(y) + \sum_{j=1}^{n} y_{k-r,j} &= \sum_{j=1}^{n} y_{kj} - \sum_{j=1}^{n} y_{k-r,j} + \sum_{j=1}^{n} y_{k-r,j} \\ &= \sum_{j=1}^{n} y_{kj} \end{aligned}$$

Similarly, it is easy to see, that the system

$$\text{(7.4b')} \qquad \sum_{j=1}^{n} y_{ij} = c'_i \text{ for } i = 1, 2, \ldots, n$$

implies the system of Equations (7.4b). So the polytope $Q_{\text{lc},2}$ is also determined by the system of inequalities determined by (7.4a), (7.4b'), and (7.4c). But the matrix of this reformulation is again (as in the proof of Theorem 7.2.2) totally unimodular, whereby the corresponding polytope is integral. $\square$

For the computational solution of h-CONVEX-RECONSTRUCTION problems the formulation (7.4) is unfortunately ill suited, as the number of nonzeros in it is already $O(n * \sum_{j=1}^{n} r_j)$ for instances with 2 directions, while the formulation for non-line-convex reconstruction requires only $O(n^2)$. As $\sum_{j=1}^{n} r_j$ usually grows at $O(n^2)$ the solution times of the LP-relaxation of this formulation are large. But the observation that the constraints (7.4b) are very similar for $i-1$ and i provides a way to a sparser integer linear program. For this we need to calculate $f_i(y) - f_{i-1}(y)$. Obviously, the following holds

$$f_i(y) - f_{i-1}(y) = \sum_{j=1}^{n} y_{ij} - \sum_{j=1}^{n} y_{i-r_j,j}. \tag{7.6}$$

Notice that the constraint (7.4b) has $\sum_i c_i$ non-zeroes (up to boundary effects) while the constraint (7.6) has only $2n$ non-zeroes. Furthermore the constraint $f_1(y) = 0$ also has at most n non-zero coefficients. Hence we can reformulate the integer linear program (7.4) into

$$\sum_{i=1}^{n} y_{ij} = \begin{cases} 0 & \text{if } r_j = 0 \\ 1 & \text{otherwise} \end{cases} \quad \text{for } j = 1, 2, \ldots, n \tag{7.7a}$$

$$\sum_{\substack{1 \le j \le n \\ r_j > 0}} y_{1j} = c_1 \tag{7.7b}$$

$$\sum_{j=1}^{n} y_{ij} - \sum_{j=1}^{n} y_{i-r_j,j} = c_i - c_{i-1} \text{ for } i = 2, 3, \ldots, n, \tag{7.7c}$$

$$0 \le y_{ij} \le 1 \text{ for } i = 1, 2, \ldots, n \text{ and for } j = 1, 2, \ldots, n, \tag{7.7d}$$

and

$$y \in \{0, 1\}^{n \times n}. \tag{7.7e}$$

Again it is possible to formulate the uniqueness problem for h-convex instances.

h-CONVEX-UNIQUENESS.

Instance: *A number $n \in \mathbb{N}$ and $\hat{x} \in \{0,1\}^{n \times n}$.*

Output: *An $\bar{x} \in \{0,1\}^{n \times n}$ fulfilling for $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, n$ the constraints $\sum_{i=1}^{n} x_{ij} = \sum_{i=1}^{n} \hat{x}_{ij}$ and $\sum_{j=1}^{n} x_{ij} = \sum_{j=1}^{n} \hat{x}_{ij}$ so that for every j the vector $(\bar{x}_{ij})_{i=1}^{n}$ has the strict consecutive ones property and $\bar{x} \neq \hat{x}$.*

This problem has a particularly easy formulation after the given solution is transformed into $\hat{y}$-variables and r and c are computed.

$$\max \quad \sum_{i=1}^{n}\sum_{j=1}^{n} y_{ij} \sum_{k=1}^{n} |i-k|\hat{y}_{kj}$$

$$\text{(7.8a)} \qquad \sum_{i=1}^{n} y_{ij} = \begin{cases} 0 & \text{if } r_j = 0 \\ 1 & \text{otherwise} \end{cases} \quad \text{for } j = 1, 2, \ldots, n,$$

$$\text{(7.8b)} \qquad \sum_{j=1}^{n} \sum_{k=i-(r_j-1)}^{i} y_{kj} = c_i \text{ for } i = 1, 2, \ldots, n,$$

$$\text{(7.8c)} \qquad 0 \leq y_{ij} \leq 1 \text{ for } i = 1, 2, \ldots, n \text{ and for } j = 1, 2, \ldots, n,$$

and

$$\text{(7.8d)} \qquad y \in \{0, 1\}^{n \times n}.$$

7.3. Comparison of the Models

To do a comparison, we implemented programs in C++ that solve problems (7.1) and (7.8). These programs are based on our class-library for problems of discrete tomography and on CPLEX 6.5 [ILO97]. The first program computes, for a given configuration a most different solution by solving the integer linear program (7.1). The other program computes for a given h-convex configuration a most different h-convex solution by solving the integer linear program (7.8).

In Figure 7.1, the frequencies of maximum differences are plotted for problems of size 20×20 with density 50%. We plot these data for $2, 3, 4, 5$ directions, in the cases of:

1. arbitrary reconstruction of an arbitrary configuration,
2. arbitrary reconstruction of an h-convex configuration, and
3. h-convex reconstruction of an h-convex configuration.

Not surprisingly, the symmetric difference for case 1 is very high, but it decreases as the number of directions grows. The picture is already surprisingly good in case 2 in the sense that most reconstructions differ only in a few places. Finally it turns out that for h-convex reconstruction of h-convex configurations (case 3) with at least 3 directions unique reconstruction is almost always guaranteed.

In Figure 7.2 and 7.3 similar results are plotted for configurations of size 40×40 with three directions and densities from 10% to 90%. It is noticeable—though not surprising—that the plots of arbitrary reconstruction of an arbitrary configuration are the same for densities p and $1 - p$.

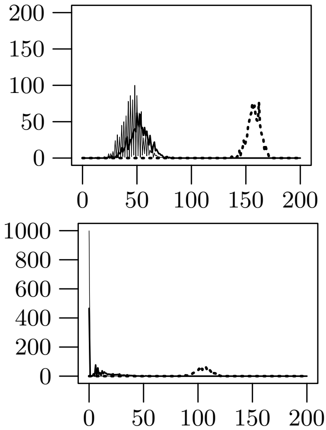


FIGURE 7.1. Frequency of maximum symmetric difference (divided by 2) for each of 1000 configurations of density 50% on a 20×20 grid with 2 (*top left*), 3 (*top right*), 4 (*bottom left*), and 5 (*bottom right*) directions for random instances and general reconstruction (⋰), for random line-convex instances and general reconstruction (/), and for random line-convex instances and line-convex reconstruction (/).

For h-convex configurations of at least 30% the h-convex reconstruction is most of the time almost unique.

It is not clear yet whether the assumption of line-convexity is fulfilled in the application to study the surface of silicon wafers. But given our previous observations, we think that it is worthwhile—if the assumption of line-convexity is fulfilled for applications—to study the algorithmic problem of exact h-convex reconstruction more so as to have a method at hand that (for many parameter settings) permits almost unique reconstruction. We also think almost-uniqueness of a reconstruction might be a very important feature in practice.

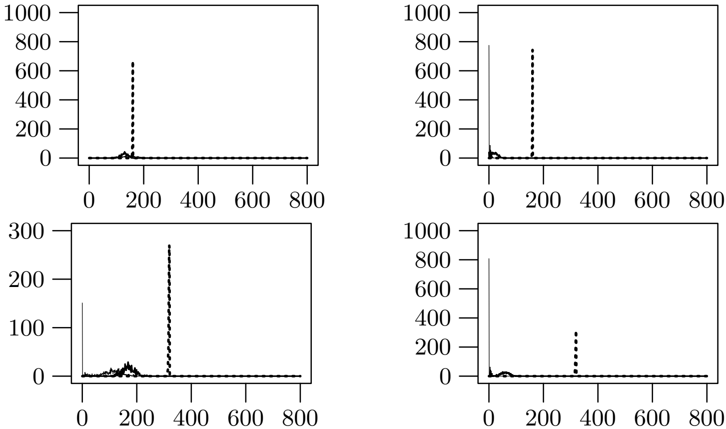


Figure 7.2. Frequency of maximum symmetric difference (divided by 2) for each of 1000 instances of density 10% (*top left*), 90% (*top right*), 20% (*bottom left*), and 80% (*bottom right*) on a 40×40 grid with 3 directions for random instances and general reconstruction (⋰), for random line-convex instances and general reconstruction (⁄), and for random line-convex instances and line-convex reconstruction (⁄).

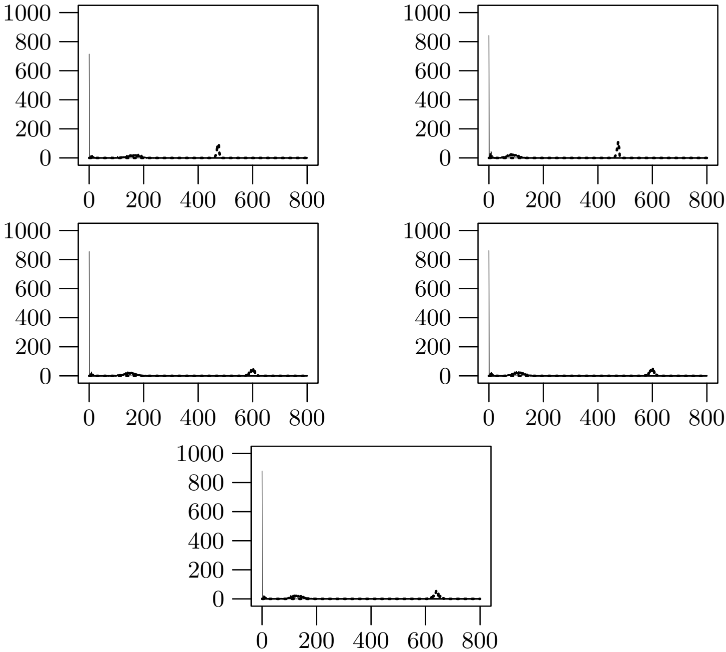


Figure 7.3. Frequency of maximum symmetric difference (divided by 2) for each of 1000 instances of density 30% (*top left*), 70% (*top right*), 40% (*middle left*), and 60% (*middle right*) and 50% (*bottom*) on a 40×40 grid with 3 directions for random instances and general reconstruction (⋰), for random line-convex instances and general reconstruction (╱), and for random line-convex instances and line-convex reconstruction (╱).

APPENDIX A

Table of Symbols

.	"The familiar dot '.' symbol from Internet addresses is used in this book to terminate sentences." [EGR98]
$\mathbb{N}$	set of natural numbers $\mathbb{N} = \{1, 2, 3 \dots\}$
$\mathbb{N}_0$	set of the natural numbers with zero $\mathbb{N}_0 = \{0, 1, 2, \dots\}$
N_n	set of natural number up to n : $\{1, 2, \dots, n\}$.
$\mathbb{Z}$	set of integer
2^V	set of all subsets of V
$\Omega_k(M)$	$\{T \subseteq M : \lvert T\rvert = k\}$
$\mathbb{Q}$	set of rational number
$\mathbb{R}$	set of real numbers
$\mathbb{R}_+$	set of the nonnegative real numbers
$\mathbf{0}$	zero vector or matrix
$\mathbf{1}$	all ones vector or matrix
$\mathbf{I}$	identity matrix
$\lceil q \rceil$	smallest integer that is at least q
$\lfloor q \rfloor$	biggest integer that is at most q
$\dot\cup$	union of disjoint sets
$\operatorname{rank}(S)$	the size of a largest independent set in S
$\operatorname{conv}(X)$	convex hull of the members of X
$\operatorname{lin}(X)$	linear hull of the members of X
$\operatorname{aff}(X)$	affine hull of the members of X
$P(A, b, b', d, d')$	$\{x \in \mathbb{R}^n : b' \leq Ax \leq b \text{ and } d' \leq x \leq d\}$

Bibliography

[AFWZ95] N. Alon, U. Feige, A. Wigderson, and D. Zuckerman, *Derandomized graph products*, Comput. Complexity **5** (1995), 60–75.

[AHK97] R. Aharoni, G. T. Herman, and A. Kuba, *Binary vectors partially determined by linear equation systems*, Discrete Math. **171** (1997), 1–16.

[AMO93] R. K. Ahuja, T. L. Magnanti, and J. B. Orlin, *Network flows: Theory, algorithms and applications*, Prentice-Hall, Englewood Cliffs, NJ (1993).

[Ans83] R. P. Anstee, *The network flows approach for matrices with given row and column sums*, Discrete Math. **44** (1983), 125–138.

[ANS95] A. Atamtürk, G. L. Nemhauser, and M. W. P. Savelsbergh, *A combined Lagrangian, linear programming and implication heuristic for large-scale set partitioning problems*, J. Heuristics **1** (1995), 247–259.

[AOST94] R. K. Ahuja, J. B. Orlin, C. Stein, and R. E. Tarjan, *Improved algorithms for bipartite network flow*, SIAM J. Comput. **23** (1994), 906–933.

[Bab93] L. Babel, *A fast algorithm for the maximum weight clique problem*, Computing **52** (1993), 31–38.

[Bab95] L. Babai, *Automorphism groups, isomorphism, reconstruction*, in *Handbook of Combinatorics* (R. L. Graham, M. Grötschel, and L. Lovász, eds.), volume 2, Elsevier, Amsterdam (1995) 1447–1540.

[Ber70] C. Berge, *Graphes et Hypergraphes*, volume 37 of *Monographies universitaires de mathématiques*, Dunod, Paris, first edition (1970).

[BGL$^+$92] R. E. Bixby, J. W. Gregory, I. J. Lustig, R. E. Marsten, and D. F. Shanno, *Very large-scale linear programming: A case study in combining interior point and simplex methods*, Oper. Res. **40** (1992), 885–897.

[Bir46] G. Birkhoff, *Tres observaciones sobre el algebra lineal*, J. Rev. Univ. Nac. Tucuman **5** (1946), 147–151.

[BLNP96] E. Barcucci, A. Del Lungo, M. Nivat, and R. Pinzani, *Reconstructing convex polyominoes from their horizontal and vertical projections*, Theor. Comput. Sci. **155** (1996), 321–347.

[BM94] F. Barahona and A. R. Mahjoub, *Compositions of graphs and polyhedra II: Stable sets*, SIAM J. Discrete Math. **7** (1994), 359–371.

[BO97] W. Borchard-Ott, *Kristallographie*, Springer-Verlag, Berlin, 5th edition (1997).

[Bol87] E. D. Bolker, *The finite radon transform*, Contemp. Math. **63** (1987), 27–50.

[Boo89] C. De Boor, *FFT as nested multiplication with a twist*, SIAM J. Sci. Stat. Comput. **1** (1989), 173–178.

[Bor97] R. Borndörfer, *Aspects of set packing, partitioning, and covering*, Ph.D. thesis, Technische Universität Berlin (1997).

[BP72] E. Balas and M. W. Padberg, *On the set-covering problem*, Oper. Res. **20** (1972), 1153–1161.

[BP75] E. Balas and M. Padberg, *On the set-covering problem: II. an algorithm for set partitioning*, Oper. Res. **23** (1975), 74–90.

[BP76] E. Balas and M. W. Padberg, *Set partitioning: A survey*, SIAM Rev. **18** (1976), 710–760.

[BSMM99] I. N. Bronstein, K. A. Semendjajew, G. Musiol, and H. Mühlig, *Taschenbuch der Mathematik*, Verlag Harri Deutsch, 4th edition (1999).

[BTV95] D. Bertismas, C. Teo, and R. Vohra, *Nonlinear formulations and improved randomized approximation algorithms for multicut problems*, in *Proc. of the Fourth International Integer Programming and Combinatorial Optimization Conference (Copenhagen/Denmark)* (E. Balas and J. Clausen, eds.), volume 920 of *LNCS*, Springer, Berlin/Heidelberg (1995) 29–39.

[BW97] R. Borndörfer and R. Weismantel, *Set packing relaxations of some integer programs*, Technical Report 97-30, Konrad-Zuse-Zentrum (1997).

[BWE92] F. Barahona, A. Weintraub, and R. Epstein, *Habitat dispersion in forest planning and the stable set problem*, Oper. Res. **40** (1992), S14–S21.

[CC95] M. Conforti and G. Cornuéjols, *Balanced* $0, \pm 1$*-matrices, bicoloring and total dual integrality*, Math. Program. **71** (1995), 249–258.

[CC97] E. Cheng and W. H. Cunningham, *Wheel inequalities for stable set polytopes*, Math. Program. **77** (1997), 389–421.

[CF96] A. Caprara and M. Fischetti, $\{0, \frac{1}{2}\}$*-Chvátal-Gomory cuts*, Math. Program. **74** (1996), 221–235.

[Cha71] S. K. Chang, *The reconstruction of binary patterns from their projections*, Commun. ACM **14** (1971), 21–25.

[Che95] E. Cheng, *Wheel inequalities for stable set polytopes*, Ph.D. thesis, University of Waterloo (1995).

[Che98] —, *Separating subdivision of bicycle wheel inequalities over cut polytopes*, Oper. Res. Lett. **23** (1998), 13–19.

[Chv75] V. Chvátal, *On certain polytopes associated with graphs*, J. Comb. Theory, Ser. B **18** (1975), 138–154.

[CK98] P. Crescenzi and V. Kann, *A compendium of NP optimization problems* (1998), URL `http://www.nada.kth.se/theory/compendium/`, preprint.

[CL88] M. Conforti and M. Laurent, *On the facial structure of independence system polyhedra*, Math. Oper. Res. **13** (1988), 543–555.

[CL89] —, *On the geometric structure of independence systems*, Math. Program. **45** (1989), 255–277.

[CM57] J. M. Cowley and A. F. Moodie, *The scattering of electrons by atoms and crystals. 1. A new theoretical approach*, Acta Cryst. **10** (1957), 609–619.

[CM58] —, *A new formulation of scalar diffraction theory for restricted aperture*, Proc. Phys. Soc. **71** (1958), 533–545.

[CT65] J. W. Cooley and J. W. Tukey, *An algorithm for the machine calculation of complex fourier series*, Math. Comput. **19** (1965), 297–301.

[Cun82] W. H. Cunningham, *Polyhedra for composed independence systems*, Ann. Discrete Math. **16** (1982), 57–67.

[CY92] T. J. Chan and C. A. Yano, *A multiplier adjustment approach for the set partitioning problem*, Oper. Res. **40** (1992), 40–47.

[Dah99] G. Dahl, *Stable set polytopes for a class of circulant graphs*, SIAM J. Optim. **9** (1999), 493–503.

[Edm79] J. Edmonds, *Matroid intersection*, Ann. Discrete Math. **4** (1979), 39–49.

[Egr98] C. Egremont III, *Mr. Bunny's Guide to ActiveX*, Addison-Wesley (1998).

[EJR87] R. Euler, M. Jünger, and G. Reinelt, *Generalizations of cliques, odd cycles and anticycles and their relation to independence system polyhedra*, Math. Oper. Res. **12** (1987), 451–462.

[EKK85] V. A. Emeličev, M. M. Kovalev, and M. K. Kravcov, *Polyeder, Graphen, Optimierung*, VEB Deutscher Verlag der Wissenschaften, Berlin (1985).

[FSSV97] P. Fishburn, P. Schwander, L. Shepp, and R. J. Vanderbei, *The discrete Radon transform and its approximate inversion via linear programming*, Discrete Appl. Math. **75** (1997), 39–61.

[Gar95] R. Gardner, *Geometric Tomography*, volume 58 of *Encyclopedia of Mathematics and its Applications*, Cambridge University Press, Cambridge (1995).

[GG97] R. J. Gardner and P. Gritzmann, *Discrete tomography: determination of finite sets by X-rays*, Trans. Am. Math. Soc. **349** (1997), 2271–2295.

[GGP96] R. J. Gardner, P. Gritzmann, and D. Prangenberg, *On the reconstruction of binary images from their discrete radon transform*, in *Proc. Intern. Symp. Optical Science, Engineering, and Instrumentation*, SPIE (1996) 121–132.

[GGP99] —, *On the computational complexity of reconstructing lattice sets from their X-rays*, Discrete Math. **202** (1999), 45–71.

[GH62] A. Ghouila-Houri, *Caractérisation des matrices totalement unimodulaires*, C. R. Acad. Sci., Paris **254** (1962), 1192–1194.

[GH88] M. C. Golumbic and P. L. Hammer, *Stability in circular arc graphs*, J. Algorithms **9** (1988), 314–320.

[Gie63] O. Giering, *Bestimmung von Eibereichen und Eikörpern durch Steiner-Symmetrisierungen*, Bayer. Akad. Wiss., Math.-naturw. Kl., Sitzungsber. (63), 225–253.

[GJ79] M. R. Garey and D. S. Johnson, *Computers and intractability: A guide to the theory of NP-completeness*, Freeman, New York (1979).

[GLS81] M. Grötschel, L. Lovász, and A. Schrijver, *The ellipsoid method and its consequences in combinatorial optimization*, Combinatorica **1** (1981), 169–197, corrigendum in: Combinatorica, **4** (1984), 291–295.

[GLS93] —, *Geometric algorithms and combinatorial optimization*, volume 2 of *Algorithms and Combinatorics*, Springer, Berlin/Heidelberg, 2nd edition (1993).

[GM74] P. Goodman and A. F. Moodie, *Numerical evalutation of N-beam wave functions in electron scattering by the multi-slice method*, Acta Cryst., Ser. A **30** (1974), 280–290.

[Goo58] I. J. Good, *The interaction algorithm and practical fourier analysis*, J. R. Stat. Soc., Ser. B **20** (1958), 361–372, addendum in: J. R. Stat. Soc., Ser. B **22** (1960), 372-375.

[GP81] M. Grötschel and W. R. Pulleyblank, *Weakly bipartite graphs and the max-cut problem*, Oper. Res. Lett. **1** (1981), 23–27.

[GP85] M. Grötschel and M. W. Padberg, *Polyhedral theory*, in *The Traveling Salesman Problem* (E. L. Lawler, J. K. Lenstra, A. H. G. Rinnoy-Kan, and D. B. Shmoys, eds.), Wiley-Interscience series in discrete mathematics and optimization, John Wiley & Sons, Chichester (1985) 251–305.

[GPVW98] P. Gritzmann, D. Prangenberg, S. de Vries, and M. Wiegelmann, *Success and failure of certain reconstruction and uniqueness algorithms in discrete tomography*, Int. J. Imaging Syst. Technol. **9** (1998), 101–109.

[Gri97] P. Gritzmann, *On the reconstruction of finite lattice sets from their X-rays*, in *Proc. 7th Intern. Workshop, DGCI '97, Montpellier, France* (E. Ahronovitz and C. Fiorio, eds.), volume 1347 of *LNCS*, Springer (1997) 19–32.

[GVW] P. Gritzmann, S. de Vries, and M. Wiegelmann, *Approximating binary images from discrete X-rays*, to appear in SIAM J. Optim.

[GW] P. Gritzmann and M. Wiegelmann, *Uniqueness and additivity in discrete tomography*, in preparation.

[Hal96] M. M. Halldórsson, *Approximating k-set cover and complementary graph coloring*, in *Proc. of the Fifth International Integer Programming and Combinatorial Optimization Conference (Vancouver/Canada)* (W. H. Cunningham, S. T. McCormick, and M. Queyranne, eds.), volume 1084 of *LNCS*, Springer-Verlag, Berlin/Heidelberg (1996) 118–131.

[Heu90] H. Heuser, *Lehrbuch der Analysis*, volume 1, Teubner, Stuttgart, 7th edition (1990).

[HJP75] P. L. Hammer, E. L. Johnson, and U. N. Peled, *Facet of regular 0-1 polytopes*, Math. Program. **8** (1975), 179–206.

[HP93] K. L. Hoffman and M. Padberg, *Solving airline crew scheduling problems by branch-and-cut*, Manage. Sci. **39** (1993), 657–682.

[HS89] C. A. J. Hurkens and A. Schrijver, *On the size of systems of sets every t of which have an SDR, with an application to the worst-case ratio of heuristic problems*, SIAM J. Discrete Math. **2** (1989), 68–72.

[ILO97] ILOG Inc. CPLEX Division, 930 Tahoe Blvd. # 802-279, Incline Village, NV 89451-9436, USA, *Using the CPLEX Callable Library* (1997), information available at URL `http://www.cplex.com`.

[IU77] K. Ishizuka and N. Uyeda, *A new theoretical and practical approach to the multislice method*, Acta Cryst. **33** (1977), 740–749.

[JT98] M. Jünger and S. Thienel, *Introduction to ABACUS—a branch-and-cut system*, Oper. Res. Lett. **23** (1998), 83–95.

[Jun90] D. Jungnickel, *Graphen, Netzwerke und Algorithmen*, BI-Wiss.-Verl., Mannheim, 2nd edition (1990).

[Kar72] R. M. Karp, *Reducibility among combinatorial problems*, in *Symposium on the Complexity of Computer Computations* (R. E. Miller and J. W. Thatcher, eds.), Plenum Press, Yorktown (1972) 85–103.

[KH78] B. Korte and D. Hausmann, *An analysis of the greedy heuristic for independence systems*, Ann. Discrete Math. **2** (1978), 65–74.

[Knu80] D. E. Knuth, *Seminumerical Algorithms*, volume 2 of *The Art of Computer Programming*, Addison-Wesley, Reading, MA, 2nd edition (1980).

[Knu94] —, *The sandwich theorem*, Electron. J. Comb. **1** (1994), A1, internet: `http://www.combinatorics.org/`.

[KSB+95] C. Kisielowski, P. Schwander, F. H. Baumann, M. Seibt, Y. Kim, and A. Ourmazd, *An approach to quantitative high-resolution transmission electron microscopy of crystalline materials*, Ultramicroscopy **58** (1995), 131–155.

[Lau89] M. Laurent, *A generalization of antiwebs to independence systems and their canonical facets*, Math. Program. **45** (1989), 97–108.

[Law75] E. L. Lawler, *Matroid intersection algorithms*, Math. Program. **9** (1975), 31–56.

[Lor49] G. G. Lorentz, *A problem of plane measure*, Amer. J. Math. **71** (1949), 417–426.

[Lov79] L. Lovász, *On the Shannon capacity of a graph*, IEEE Trans. Inf. Theory **25** (1979), 1–7.

[Lov86] L. Lovász, *An algorithmic theory of numbers, graphs and convexity*, CBMS-NSF Reg. Conf. Ser. Appl. Math., SIAM (1986).

[LPR97] M. Laurent, S. Poljak, and F. Rendl, *Connections between semidefinite relaxations of the max-cut and stable set problems*, Math. Program. **77** (1997), 225–246.

[LRK79] J. K. Lenstra and A. H. G. Rinnoy-Kan, *Complexity of packing, covering and partitioning problems*, in *Packing and Covering in Combinatorics* (A. Schrijver, ed.), Mathematical Centre Tracts, Mathematical Centre, Amsterdam (1979) 275–291.

[Meg91] N. Megiddo, *On finding primal- and dual-optimal bases*, ORSA J. Comput. **3** (1991), 63–65.

[MHV98] S. Matej, G. T. Herman, and A. Vardi, *Binary tomography on the hexagonal grid using gibbs priors*, Int. J. Imaging Syst. Technol. **9** (1998), 126–131.

[MS96] R. Müller and A. S. Schulz, *Transitive packing*, in *Proc. of the Fifth International Integer Programming and Combinatorial Optimization Conference (Vancouver/Canada)* (W. H. Cunningham, S. T. McCormick, and M. Queyranne, eds.), volume 1084 of *LNCS*, Springer, Berlin/Heidelberg (1996) 430–444.

[Neu53] J. von Neumann, *A certain zero-sum two-person game equivalent to the optimal assignment problem*, Ann. Math. Studies (1953), 5–12.

[NS92] G. L. Nemhauser and G. Sigismondi, *A strong cutting plane/branch-and-bound algorithm for node packing*, J. Oper. Res. Soc (1992), 443–457.

[NT74] G. L. Nemhauser and L. E. Trotter Jr., *Properties of vertex packing and independence system polyhedra*, Math. Program. **6** (1974), 48–61.

[NW88] G. L. Nemhauser and L. A. Wolsey, *Integer and combinatorial optimization*, Wiley-Interscience series in discrete mathematics and optimization, John Wiley & Sons, New York (1988).

[Pad73] M. W. Padberg, *On the facial structure of set packing polyhedra*, Math. Program. **5** (1973), 199–215.

[Pap94] C. H. Papadimitriou, *Computational Complexity*, Addison-Wesley, Reading, MA (1994).

[PR82] M. W. Padberg and M. R. Rao, *Odd minimum cut-sets and b-matchings*, Math. Oper. Res. **7** (1982), 67–80.

[PY91] C. H. Papadimitriou and M. Yannakakis, *Optimization, approximation, and complexity classes*, J. Comput. System Sci. **43** (1991), 425–440.

[Rad17] J. Radon, *Über die Bestimmung von Funktionen durch ihre Integralwerte längs gewisser Mannigfaltigkeiten*, Berichte Sächsische Akademie der Wissenschaften, Math.-Phys. Kl. **69** (1917), 262–267.

[Rys57] H. J. Ryser, *Combinatorial properties of matrices of zeros and ones*, Canad. J. Math. **9** (1957), 371–377.

[Rys63] —, *Combinatorial Mathematics*, Mathematical Association of America and Quinn & Boden, Rahway, NJ (1963).

[Sch89] A. Schrijver, *Theory of Linear and Integer Programming*, Wiley-Interscience series in discrete mathematics and optimization, John Wiley & Sons, Chichester (1989).

[Sch96] A. S. Schulz, *Polytopes and scheduling*, Ph.D. thesis, TU Berlin (1996).

[Sek83] Y. Sekiguchi, *A note on node packing polytopes on hypergraphs*, Oper. Res. Lett. **2** (1983), 143–247.

[SG82] C. H. Slump and J. J. Gerbrands, *A network flow approach to reconstruction of the left ventricle from two projections*, Comput. Graphics and Image Process. **18** (1982), 18–36.

[Sim90] C. De Simone, *The cut polytope and the boolean quadric polytope*, Discrete Math. **79** (1989/90), 71–75.

[SKB+93] P. Schwander, C. Kisielowski, F. H. Baumann, Y. Kim, and A. Ourmazd, *Mapping projected potential, interfacial roughness, and composition in general crystalline solids by quantitative transmission electron microscopy*, Physical Review Letters **71** (1993), 4150–4153.

[SRVR98] P. M. Salzberg, P. I. Rivera-Vega, and A. Rodríguez, *A network flow model for binary tomography on lattices*, Int. J. Imaging Syst. Technol. **9** (1998), 147–154.

[Sta87] P. A. Stadelmann, *EMS—a software package for electron diffraction analysis and HREM image simulation in material science*, Ultramicroscopy **21** (1987), 131–146.

[SWZ95] A. S. Schulz, R. Weismantel, and G. M. Ziegler, *0/1-integer programming: Optimization and augmentation are equivalent*, in *Algorithms-ESA '95 (Corfu/Greece)* (P. Spirakis, ed.), volume 979 of *LNCS*, Springer, Berlin/Heidelberg (1995) 473–483.

[Tes94] D. Tesch, *Disposition von Anruf-Sammeltaxis*, Ph.D. thesis, Universität Passau (1994).

[Tho98] M. Thorup, *Floats, integers, and single source shortest paths*, in *Proceedings of the 15th Annual Symposium on Theoretical Aspects of Computer Science (STACS)* (M. Morvan, C. Meinel, and D. Krobs, eds.), volume 1373 of *LNCS*, Springer, Berlin/Heidelberg (1998) 14–24.

[Tro75] L. E. Trotter, *A class of facet producing graphs for vertex packing polyhedra*, Discrete Math. **12** (1975), 373–388.

[Vla86] M. Vlach, *Conditions for the existence of solutions of the three-dimensional planar transportation problem*, Discrete Appl. Math. **13** (1986), 61–78.

[Woe96] G. J. Woeginger, *The Reconstruction of Polyminoes from their Orthogonal Projections*, Technical report, Karl-Franzens-Universität Graz & Technische Universität Graz (1996).

[Wol76] L. A. Wolsey, *Further facet generating procedures for vertex packing polytopes*, Math. Program. **11** (1976), 158–163.

[Zie95] G. M. Ziegler, *Lectures on Polytopes*, volume 152 of *Graduate texts in mathematics*, Springer, New York, 2nd edition (1995).

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